Chen Shuwen's Notebooks (Part I)

陈漱文笔记（第一卷）

（1st Edition，2021/8/16）

Chapter 1. The Generalization of Ramanujan nested radicals

$$2\cos\frac{355\pi}{113} = -\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-2\cos\frac{355\pi}{113}}}}}}}}}}}}}}}$$

Chapter 2. The Generalization of Euler's formula

$$\left(\cos\frac{0\pi}{11}\right)^k+\left(\cos\frac{2\pi}{11}\right)^k+\left(\cos\frac{2\pi}{11}\right)^k+\left(\cos\frac{4\pi}{11}\right)^k+\left(\cos\frac{4\pi}{11}\right)^k+\left(\cos\frac{6\pi}{11}\right)^k+\left(\cos\frac{6\pi}{11}\right)^k+\left(\cos\frac{8\pi}{11}\right)^k+\left(\cos\frac{8\pi}{11}\right)^k+\left(\cos\frac{10\pi}{11}\right)^k+\left(\cos\frac{10\pi}{11}\right)^k$$

$$=\left(\cos\frac{1\pi}{11}\right)^k+\left(\cos\frac{1\pi}{11}\right)^k+\left(\cos\frac{3\pi}{11}\right)^k+\left(\cos\frac{3\pi}{11}\right)^k+\left(\cos\frac{5\pi}{11}\right)^k+\left(\cos\frac{5\pi}{11}\right)^k+\left(\cos\frac{7\pi}{11}\right)^k+\left(\cos\frac{7\pi}{11}\right)^k+\left(\cos\frac{9\pi}{11}\right)^k+\left(\cos\frac{9\pi}{11}\right)^k+\left(\cos\frac{11\pi}{11}\right)^k$$

$$(\ k = 1,2,3,4,5,6,7,8,9,10\ )$$

Chapter 3. The Generalization of Newton Identities

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k \quad (\ k = 1,3,5,7,9\ )$$
$$G_1 = -P_1$$
$$G_3 = (P_1^3 - P_3)/3$$
$$G_5 = (P_1^5 - P_5)/5$$
$$G_7 = (P_1^7 - P_7)/7$$
$$G_9 = (P_1^9 - P_9)/9 - (G_3^3/3)$$

$$\Rightarrow \quad G_1^2 + 2\frac{\begin{vmatrix} G_5 & 0 & 1 \\ G_7 & G_3 & G_1^2 \\ G_9 & G_5 & G_1^4 + G_3G_1 \end{vmatrix}}{\begin{vmatrix} G_3 & 0 & 1 \\ G_5 & G_3 & G_1^2 \\ G_7 & G_5 & G_1^4 + G_3G_1 \end{vmatrix}} = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2$$

Chapter 4. The Generalization of Ramanujan Identity

$$P_k = a_1^k + a_2^k + a_3^k - b_1^k - b_2^k - b_3^k \quad (\ k = 1,2,3,4,5,6\ )$$
$$S_1 = (P_1)/1$$
$$S_2 = (P_2 + S_1P_1)/2$$
$$S_3 = (P_3 + S_1P_2 + S_2P_1)/3$$
$$S_4 = (P_4 + S_1P_3 + S_2P_2 + S_3P_1)/4$$
$$S_5 = (P_5 + S_1P_4 + S_2P_3 + S_3P_2 + S_4P_1)/5$$
$$S_6 = (P_6 + S_1P_5 + S_2P_4 + S_3P_3 + S_4P_2 + S_5P_1)/6$$

$$\Rightarrow \quad \frac{\begin{vmatrix} S_3 & S_2 & S_1 \\ S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 \end{vmatrix}}{\begin{vmatrix} S_3 & S_2 & S_1 \\ S_4 & S_3 & S_2 \\ S_5 & S_4 & S_3 \end{vmatrix}} = a_1 + a_2 + a_3$$

Chapter 5. The Generalization of the Prouhet-Tarry-Escott Problem

$$22\times22\times33\times33\times56\times56\times84\times84 = 21\times24\times28\times42\times44\times66\times77\times88$$

$$22^k+22^k+33^k+33^k+56^k+56^k+84^k+84^k = 21^k+24^k+28^k+42^k+44^k+66^k+77^k+88^k$$

$$(\ k = -3,-2,-1,1,2,3\ )$$

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Chapter 1

The Generalization of Ramanujan nested radicals

Theorem 1.1. (Chen Shuwen, 2021/7/25)

Let p_i be odd primes, k_i be positive integers. Then

(1) for any integer $h > 1$,

$$2^{\left(2\prod_{i=1}^{h}\frac{p_i-1}{2}p_i^{k_i-1}\right)} - 1 \equiv 0\left(mod \prod_{i=1}^{h} p_i^{k_i}\right)$$

(2) for $p_1 \equiv \pm 1(mod\ 8)$,

$$2^{\left(\frac{p_1-1}{2}p_1^{k_1-1}\right)} - 1 \equiv 0\left(mod\ p_1^{k_1}\right)$$

(3) for $p_1 \equiv \pm 3(mod\ 8)$,

$$2^{\left(\frac{p_1-1}{2}p_1^{k_1-1}\right)} + 1 \equiv 0\left(mod\ p_1^{k_1}\right)$$

Example 1.1.

$$2^{2\left(\frac{7-1}{2}*7^0\right)\left(\frac{1031}{2}*1031^1\right)\left(\frac{19-1}{2}*19^2\right)} - 1 \equiv 0\left(mod(7 * 19^3 * 1031^2)\right)$$

$$2^{2(18*41)} - 1 \equiv 0(mod(37 * 83))$$

$$2^{8*17} - 1 \equiv 0\left(mod(17^2)\right)$$

$$2^{3*7^3} - 1 \equiv 0\left(mod(7^4)\right)$$

$$2^{9*19} + 1 \equiv 0\left(mod(19^2)\right)$$

$$2^{26} + 1 \equiv 0\left(mod(53)\right)$$

$$2^{26*53^3} + 1 \equiv 0(mod(53^4))$$

Theorem 1.2. (Chen Shuwen, 2021/7/22, proved by using Theorem 1.1)

For any rational number $\frac{n}{m}$, both $\sin(\frac{n\pi}{m})$ and $\cos(\frac{n\pi}{m})$ can be represented as cyclic infinite nested square roots of 2, of which the cyclic period is less than $\frac{m-1}{2}$.

Example 1.2.

$$2\sin\left(\frac{\pi}{18}\right) = \sqrt{2-\sqrt{2+\sqrt{2+2\sin\left(\frac{\pi}{18}\right)}}} \qquad \text{(Ramanujan)}$$

$$2\cos\left(\frac{\pi}{11}\right)=\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-2\cos\left(\frac{\pi}{11}\right)}}}}}$$ (Sivakumar Krishnamoorthi)

Example 1.3. (Chen Shuwen, July 2021)

$$2\cos 0=\sqrt{2+2\cos 0}$$

$$2\cos\frac{1\pi}{3}=\sqrt{2-2\cos\frac{1\pi}{3}}$$

$$2\cos\frac{1\pi}{5}=\sqrt{2+\sqrt{2-2\cos\frac{1\pi}{5}}}$$

$$2\cos\frac{\pi}{7}=\sqrt{2+\sqrt{2-\sqrt{2-2\cos\frac{\pi}{7}}}}$$

$$2\cos\frac{\pi}{9}=\sqrt{2+\sqrt{2+\sqrt{2-2\cos\frac{\pi}{9}}}}$$

$$2\cos\frac{\pi}{13}=\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-2\cos\frac{\pi}{13}}}}}}}$$

$$2\cos\frac{\pi}{15}=\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-2\cos\frac{\pi}{15}}}}}$$

$$2\cos\frac{\pi}{17}=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-2\cos\frac{\pi}{17}}}}}$$

$$2\cos\frac{\pi}{19}=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-2\cos\frac{\pi}{19}}}}}}}}}}$$

$$2\cos\frac{\pi}{21}=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-2\cos\frac{\pi}{21}}}}}}}$$

$$2\cos\frac{\pi}{23}=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-2\cos\frac{\pi}{23}}}}}}}}}}}$$

$$2\cos\frac{\pi}{25}=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2-2\cos\frac{\pi}{25}}}}}}}}}}$$

$$2\cos\frac{\pi}{27}=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-2\cos\frac{\pi}{27}}}}}}}}}$$

$$2\cos\frac{\pi}{29}=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-2\cos\frac{\pi}{29}}}}}}}}}}}}}}$$

$$2\cos\frac{\pi}{31}=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-2\cos\frac{\pi}{31}}}}}}$$

$$2\cos\frac{\pi}{33}=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\cos\frac{\pi}{33}}}}}}$$

$$2\cos\frac{\pi}{35}=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-2\cos\frac{\pi}{35}}}}}}}}}}}}$$

$$2\cos\frac{\pi}{37}=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-2\cos\frac{\pi}{37}}}}}}}}}}}}}}}}}}$$

$$2\cos\frac{\pi}{39}=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-2\cos\frac{\pi}{39}}}}}}}}}}}}}$$

$$2\cos\frac{\pi}{41}=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2-2\cos\frac{\pi}{41}}}}}}}}}}$$

$$2\cos\frac{\pi}{43}=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-2\cos\frac{\pi}{43}}}}}}}}$$

$$2\cos\frac{\pi}{45}=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-2\cos\frac{\pi}{45}}}}}}}}}}}}$$

$$2\cos\frac{\pi}{47}=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-2\cos\frac{\pi}{47}}}}}}}}}}}}}}}}}}}}}}}$$

$$2\cos\frac{\pi}{49}=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-2\cos\frac{\pi}{49}}}}}}}}}}}}}}}}}}}}}$$

Example 1.4. (Chen Shuwen, July 2021)

$$2\cos\frac{\pi}{13}=\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-2\cos\frac{\pi}{13}}}}}}}$$

$$2\cos\frac{2\pi}{13}=\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+2\cos\frac{2\pi}{13}}}}}}}$$

$$2\sin\frac{\pi}{13}=\sqrt{2-2\cos\frac{2\pi}{13}}$$

$$2\cos\frac{\pi}{26} = \sqrt{2 + 2\cos\frac{\pi}{13}}$$

$$2\cos\frac{\pi}{52} = \sqrt{2 + \sqrt{2 + 2\cos\frac{\pi}{13}}}$$

$$2\cos\frac{\pi}{104} = \sqrt{2 + \sqrt{2 + \sqrt{2 + 2\cos\frac{\pi}{13}}}}$$

$$2\sin\frac{\pi}{104} = \sqrt{2 - \sqrt{2 + \sqrt{2 + 2\cos\frac{\pi}{13}}}}$$

$$2\sin\frac{\pi}{208} = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + 2\cos\frac{\pi}{13}}}}}$$

$$2\sin\frac{27\pi}{208} = \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 + 2\cos\frac{\pi}{13}}}}}$$

$$2\sin\frac{77\pi}{208} = \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 + 2\cos\frac{\pi}{13}}}}}$$

$$2\sin\frac{79\pi}{208} = \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 + 2\cos\frac{\pi}{13}}}}}$$

Example 1.5. (Chen Shuwen, July 2021)

$$2\cos\frac{\pi}{19} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 - 2\cos\frac{\pi}{19}}}}}}}}}}$$

$$2\cos\frac{2\pi}{19} = \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 + 2\cos\frac{2\pi}{19}}}}}}}}}}$$

$$2\cos\frac{3\pi}{19} = \sqrt{2 + \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 - 2\cos\frac{3\pi}{19}}}}}}}}}}$$

$$2\cos\frac{4\pi}{19}=\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2+2\cos\frac{4\pi}{19}}}}}}}}}}$$

$$2\cos\frac{5\pi}{19}=\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-2\cos\frac{5\pi}{19}}}}}}}}}}$$

$$2\cos\frac{6\pi}{19}=\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2+2\cos\frac{6\pi}{19}}}}}}}}}}$$

$$2\cos\frac{7\pi}{19}=\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-2\cos\frac{7\pi}{19}}}}}}}}}}$$

$$2\cos\frac{8\pi}{19}=\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+2\cos\frac{8\pi}{19}}}}}}}}}}$$

$$2\cos\frac{9\pi}{19}=\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-2\cos\frac{9\pi}{19}}}}}}}}}}$$

Example 1.6. (Chen Shuwen, August 2021)

$$2\cos\frac{355\pi}{113}=-\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-2\cos\frac{355\pi}{113}}}}}}}}}}}}}}}$$

$$2\cos\frac{\pi}{65537}=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-2\cos\frac{\pi}{65537}}}}}}}}}}}}}}}}}$$

$$2\cos\frac{\pi}{1729}=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-2\cos\frac{\pi}{1729}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}$$

$$2\cos\frac{\pi}{641}=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2-2\cos\frac{\pi}{641}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}}$$

Identities 1.1. (Chen Shuwen, 2021/8/5)

$$\sqrt{1+0\sqrt{1+1\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+6\sqrt{1+7\sqrt{1+8\sqrt{1+9\sqrt{1+10\sqrt{1+\cdots}}}}}}}}}}}}=1$$

$$\sqrt{1+1\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+6\sqrt{1+7\sqrt{1+8\sqrt{1+9\sqrt{1+10\sqrt{1+\cdots}}}}}}}}}}}=2$$

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+6\sqrt{1+7\sqrt{1+8\sqrt{1+9\sqrt{1+10\sqrt{1+\cdots}}}}}}}}}}=3 \qquad \text{(Ramanujan)}$$

$$\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+6\sqrt{1+7\sqrt{1+8\sqrt{1+9\sqrt{1+10\sqrt{1+\cdots}}}}}}}}}=4$$

$$\sqrt{1+4\sqrt{1+5\sqrt{1+6\sqrt{1+7\sqrt{1+8\sqrt{1+9\sqrt{1+10\sqrt{1+\cdots}}}}}}}}=5$$

… …

Identities 1.2. (Chen Shuwen, 2021/8/5)

$$\sqrt[3]{1+0\times 1\sqrt[3]{2+1\times 2\sqrt[3]{3+2\times 3\sqrt[3]{4+3\times 4\sqrt[3]{5+4\times 5\sqrt[3]{6+5\times 6\sqrt[3]{7+6\times 7\sqrt[3]{8+7\times 8\sqrt[3]{9+\cdots}}}}}}}}=1$$

$$\sqrt[3]{2+1\times 2\sqrt[3]{3+2\times 3\sqrt[3]{4+3\times 4\sqrt[3]{5+4\times 5\sqrt[3]{6+5\times 6\sqrt[3]{7+6\times 7\sqrt[3]{8+7\times 8\sqrt[3]{9+\cdots}}}}}}}=2$$

$$\sqrt[3]{3+2\times 3\sqrt[3]{4+3\times 4\sqrt[3]{5+4\times 5\sqrt[3]{6+5\times 6\sqrt[3]{7+6\times 7\sqrt[3]{8+7\times 8\sqrt[3]{9+\cdots}}}}}}=3$$

$$\sqrt[3]{4+3\times 4\sqrt[3]{5+4\times 5\sqrt[3]{6+5\times 6\sqrt[3]{7+6\times 7\sqrt[3]{8+7\times 8\sqrt[3]{9+\cdots}}}}}=4$$

$$\sqrt[3]{5+4\times 5\sqrt[3]{6+5\times 6\sqrt[3]{7+6\times 7\sqrt[3]{8+7\times 8\sqrt[3]{9+\cdots}}}}=5$$

…… ……

Reference 1.1.

Chapter 2

The Generalization of Euler's formula

Formula 2.1. (Chen Shuwen, 2021/7/28)

$$\sum_{k=1}^{n} e^{(-1)^{k+1}\frac{ki\pi}{n}} + 1 = 0$$

(n is odd integer and $n \geq 1$)

Example 2.1. (Chen Shuwen, 2021/7/28)

$$e^{\frac{1i\pi}{1}} + 1 = 0 \qquad \text{(Euler)}$$

$$e^{\frac{1i\pi}{3}} + e^{-\frac{2i\pi}{3}} + e^{\frac{3i\pi}{3}} + 1 = 0$$

$$e^{\frac{1i\pi}{5}} + e^{-\frac{2i\pi}{5}} + e^{\frac{3i\pi}{5}} + e^{-\frac{4i\pi}{5}} + e^{\frac{5i\pi}{5}} + 1 = 0$$

$$e^{\frac{1i\pi}{7}} + e^{-\frac{2i\pi}{7}} + e^{\frac{3i\pi}{7}} + e^{-\frac{4i\pi}{7}} + e^{\frac{5i\pi}{7}} + e^{-\frac{6i\pi}{7}} + e^{\frac{7i\pi}{7}} + 1 = 0$$

$$e^{\frac{1i\pi}{9}} + e^{-\frac{2i\pi}{9}} + e^{\frac{3i\pi}{9}} + e^{-\frac{4i\pi}{9}} + e^{\frac{5i\pi}{9}} + e^{-\frac{6i\pi}{9}} + e^{\frac{7i\pi}{9}} + e^{-\frac{8i\pi}{9}} + e^{\frac{9i\pi}{9}} + 1 = 0$$

$$e^{\frac{i\pi}{11}} + e^{-\frac{2i\pi}{11}} + e^{\frac{3i\pi}{11}} + e^{-\frac{4i\pi}{11}} + e^{\frac{5i\pi}{11}} + e^{-\frac{6i\pi}{11}} + e^{\frac{7i\pi}{11}} + e^{-\frac{8i\pi}{11}} + e^{\frac{9i\pi}{11}} + e^{-\frac{10i\pi}{11}} + e^{\frac{11i\pi}{11}} + 1 = 0$$

Formula 2.1.1. (Simplification of Formula 2.1. Chen Shuwen, 2021/7/28)

$$\sum_{k=1}^{n-1} e^{(-1)^{k+1}\frac{ki\pi}{n}} = 0$$

(n is odd integer and $n \geq 3$)

Example 2.1.1

$$e^{\frac{1i\pi}{3}} + e^{-\frac{2i\pi}{3}} = 0$$

$$e^{\frac{1i\pi}{5}} + e^{-\frac{2i\pi}{5}} + e^{\frac{3i\pi}{5}} + e^{-\frac{4i\pi}{5}} = 0$$

$$e^{\frac{1i\pi}{7}} + e^{-\frac{2i\pi}{7}} + e^{\frac{3i\pi}{7}} + e^{-\frac{4i\pi}{7}} + e^{\frac{5i\pi}{7}} + e^{-\frac{6i\pi}{7}} = 0$$

$$e^{\frac{1i\pi}{9}} + e^{-\frac{2i\pi}{9}} + e^{\frac{3i\pi}{9}} + e^{-\frac{4i\pi}{9}} + e^{\frac{5i\pi}{9}} + e^{-\frac{6i\pi}{9}} + e^{\frac{7i\pi}{9}} + e^{-\frac{8i\pi}{9}} = 0$$

$$e^{\frac{i\pi}{11}} + e^{-\frac{2i\pi}{11}} + e^{\frac{3i\pi}{11}} + e^{-\frac{4i\pi}{11}} + e^{\frac{5i\pi}{11}} + e^{-\frac{6i\pi}{11}} + e^{\frac{7i\pi}{11}} + e^{-\frac{8i\pi}{11}} + e^{\frac{9i\pi}{11}} + e^{-\frac{10i\pi}{11}} = 0$$

Formula 2.2. (Chen Shuwen, 2021/7/6)

$$\sum_{k=1}^{n}(-1)^{k+1}e^{\frac{ki\pi}{n}}=0$$

(n is odd integer and $n\geq 3$)

Example 2.2. (Chen Shuwen, 2021/7/6)

$$e^{\frac{i\pi}{3}}-e^{\frac{2i\pi}{3}}+e^{\frac{3i\pi}{3}}=0$$

$$e^{\frac{i\pi}{5}}-e^{\frac{2i\pi}{5}}+e^{\frac{3i\pi}{5}}-e^{\frac{4i\pi}{5}}+e^{\frac{5i\pi}{5}}=0$$

$$e^{\frac{i\pi}{7}}-e^{\frac{2i\pi}{7}}+e^{\frac{3i\pi}{7}}-e^{\frac{4i\pi}{7}}+e^{\frac{5i\pi}{7}}-e^{\frac{6i\pi}{7}}+e^{\frac{7i\pi}{7}}=0$$

$$e^{\frac{i\pi}{9}}-e^{\frac{2i\pi}{9}}+e^{\frac{3i\pi}{9}}-e^{\frac{4i\pi}{9}}+e^{\frac{5i\pi}{9}}-e^{\frac{6i\pi}{9}}+e^{\frac{7i\pi}{9}}-e^{\frac{8i\pi}{9}}+e^{\frac{9i\pi}{9}}=0$$

$$e^{\frac{i\pi}{11}}-e^{\frac{2i\pi}{11}}+e^{\frac{3i\pi}{11}}-e^{\frac{4i\pi}{11}}+e^{\frac{5i\pi}{11}}-e^{\frac{6i\pi}{11}}+e^{\frac{7i\pi}{11}}-e^{\frac{8i\pi}{11}}+e^{\frac{9i\pi}{11}}-e^{\frac{10i\pi}{11}}+e^{\frac{11i\pi}{11}}=0$$

Formula 2.2.1. (Simplification of Formula 2.2. Chen Shuwen, 2021/7/6)

$$\sum_{k=1}^{n-1}(-1)^{k+1}e^{\frac{ki\pi}{n}}=1$$

(n is odd integer and $n\geq 3$)

Example 2.2.1

$$e^{\frac{i\pi}{3}}-e^{\frac{2i\pi}{3}}=1$$

$$e^{\frac{i\pi}{5}}-e^{\frac{2i\pi}{5}}+e^{\frac{3i\pi}{5}}-e^{\frac{4i\pi}{5}}=1$$

$$e^{\frac{i\pi}{7}}-e^{\frac{2i\pi}{7}}+e^{\frac{3i\pi}{7}}-e^{\frac{4i\pi}{7}}+e^{\frac{5i\pi}{7}}-e^{\frac{6i\pi}{7}}=1$$

$$e^{\frac{i\pi}{9}}-e^{\frac{2i\pi}{9}}+e^{\frac{3i\pi}{9}}-e^{\frac{4i\pi}{9}}+e^{\frac{5i\pi}{9}}-e^{\frac{6i\pi}{9}}+e^{\frac{7i\pi}{9}}-e^{\frac{8i\pi}{9}}=1$$

Formula 2.3. (Euler, Infinite analysis Introduction)

$$\cos z+\sum_{k=1}^{\frac{n}{2}-1}(-1)^{k}\left(\cos\left(\frac{k\pi}{n}+z\right)+\cos\left(\frac{k\pi}{n}-z\right)\right)=0$$

(n is odd integer and $n\geq 3$, z is real number.)

It is equivalent to

$$\sum_{k=0}^{n-1}(-1)^{k}\cos\left(\frac{k\pi}{n}+z\right)=0$$

Example 2.3. (Euler)

$$\cos z-\cos\left(\frac{\pi}{3}+z\right)+\cos\left(\frac{2\pi}{3}+z\right)=0$$

$$\cos z-\cos\left(\frac{\pi}{5}+z\right)+\cos\left(\frac{2\pi}{5}+z\right)-\cos\left(\frac{3\pi}{5}+z\right)+\cos\left(\frac{4\pi}{5}+z\right)=0$$

$$\cos z-\cos\left(\frac{\pi}{7}+z\right)+\cos\left(\frac{2\pi}{7}+z\right)-\cos\left(\frac{3\pi}{7}+z\right)+\cos\left(\frac{4\pi}{7}+z\right)-\cos\left(\frac{5\pi}{7}+z\right)+\cos\left(\frac{6\pi}{7}+z\right)=0$$

Example 2.3.1 ($z=0$)

$$\cos\left(\frac{1\pi}{3}\right)=\frac{1}{2}$$

$$\cos\left(\frac{1\pi}{5}\right)+\cos\left(\frac{3\pi}{5}\right)=\frac{1}{2}$$

$$\cos\left(\frac{1\pi}{7}\right)+\cos\left(\frac{3\pi}{7}\right)+\cos\left(\frac{5\pi}{7}\right)=\frac{1}{2}$$

$$\cos\left(\frac{1\pi}{9}\right)+\cos\left(\frac{3\pi}{9}\right)+\cos\left(\frac{5\pi}{9}\right)+\cos\left(\frac{7\pi}{9}\right)=\frac{1}{2}$$

$$\cos\left(\frac{1\pi}{11}\right)+\cos\left(\frac{3\pi}{11}\right)+\cos\left(\frac{5\pi}{11}\right)+\cos\left(\frac{7\pi}{11}\right)+\cos\left(\frac{9\pi}{11}\right)=\frac{1}{2}$$

$$\cos\left(\frac{1\pi}{13}\right)+\cos\left(\frac{3\pi}{13}\right)+\cos\left(\frac{5\pi}{13}\right)+\cos\left(\frac{7\pi}{13}\right)+\cos\left(\frac{9\pi}{13}\right)+\cos\left(\frac{11\pi}{13}\right)=\frac{1}{2}$$

Formula 2.4. (Chen Shuwen, 2021/7/7)

$$\sum_{k=0}^{n-1}(-1)^k\left(\cos\left(\frac{k\pi}{n}+z\right)\right)^h=0$$

(n is odd integer and $n\geq 3$, z is real number. $h=1,3,\ldots,n-2$)

Example 2.4.1 (Chen Shuwen, July 2021)

$$\cos^1\left(\frac{1\pi}{7}+z\right)+\cos^1\left(\frac{3\pi}{7}+z\right)+\cos^1\left(\frac{5\pi}{7}+z\right)=\cos^1\left(\frac{0\pi}{7}+z\right)+\cos^1\left(\frac{2\pi}{7}+z\right)+\cos^1\left(\frac{4\pi}{7}+z\right)+\cos^1\left(\frac{6\pi}{7}+z\right)$$

$$\cos^3\left(\frac{1\pi}{7}+z\right)+\cos^3\left(\frac{3\pi}{7}+z\right)+\cos^3\left(\frac{5\pi}{7}+z\right)=\cos^3\left(\frac{0\pi}{7}+z\right)+\cos^3\left(\frac{2\pi}{7}+z\right)+\cos^3\left(\frac{4\pi}{7}+z\right)+\cos^3\left(\frac{6\pi}{7}+z\right)$$

$$\cos^5\left(\frac{1\pi}{7}+z\right)+\cos^5\left(\frac{3\pi}{7}+z\right)+\cos^5\left(\frac{5\pi}{7}+z\right)=\cos^5\left(\frac{0\pi}{7}+z\right)+\cos^5\left(\frac{2\pi}{7}+z\right)+\cos^5\left(\frac{4\pi}{7}+z\right)+\cos^5\left(\frac{6\pi}{7}+z\right)$$

Example 2.4.2 (Chen Shuwen, July 2021)

$$\cos^1\left(\frac{1\pi}{7}\right)+\cos^1\left(\frac{3\pi}{7}\right)+\cos^1\left(\frac{5\pi}{7}\right)=\cos^1\left(\frac{0\pi}{7}\right)+\cos^1\left(\frac{2\pi}{7}\right)+\cos^1\left(\frac{4\pi}{7}\right)+\cos^1\left(\frac{6\pi}{7}\right)=\frac{1}{2}$$

$$\cos^3\left(\frac{1\pi}{7}\right)+\cos^3\left(\frac{3\pi}{7}\right)+\cos^3\left(\frac{5\pi}{7}\right)=\cos^3\left(\frac{0\pi}{7}\right)+\cos^3\left(\frac{2\pi}{7}\right)+\cos^3\left(\frac{4\pi}{7}\right)+\cos^3\left(\frac{6\pi}{7}\right)=\frac{1}{2}$$

$$\cos^5\left(\frac{1\pi}{7}\right)+\cos^5\left(\frac{3\pi}{7}\right)+\cos^5\left(\frac{5\pi}{7}\right)=\cos^5\left(\frac{0\pi}{7}\right)+\cos^5\left(\frac{2\pi}{7}\right)+\cos^5\left(\frac{4\pi}{7}\right)+\cos^5\left(\frac{6\pi}{7}\right)=\frac{1}{2}$$

Example 2.4.3 (Chen Shuwen, July 2021)

$$\cos\left(\frac{\pi}{27}\right)^1+\cos\left(\frac{5\pi}{27}\right)^1+\cos\left(\frac{7\pi}{27}\right)^1+\cos\left(\frac{11\pi}{27}\right)^1+\cos\left(\frac{13\pi}{27}\right)^1=\cos\left(\frac{2\pi}{27}\right)^1+\cos\left(\frac{4\pi}{27}\right)^1+\cos\left(\frac{8\pi}{27}\right)^1+\cos\left(\frac{10\pi}{27}\right)^1$$

$$\cos\left(\frac{\pi}{27}\right)^3+\cos\left(\frac{5\pi}{27}\right)^3+\cos\left(\frac{7\pi}{27}\right)^3+\cos\left(\frac{11\pi}{27}\right)^3+\cos\left(\frac{13\pi}{27}\right)^3=\cos\left(\frac{2\pi}{27}\right)^3+\cos\left(\frac{4\pi}{27}\right)^3+\cos\left(\frac{8\pi}{27}\right)^3+\cos\left(\frac{10\pi}{27}\right)^3$$

$$\cos\left(\frac{\pi}{27}\right)^5+\cos\left(\frac{5\pi}{27}\right)^5+\cos\left(\frac{7\pi}{27}\right)^5+\cos\left(\frac{11\pi}{27}\right)^5+\cos\left(\frac{13\pi}{27}\right)^5=\cos\left(\frac{2\pi}{27}\right)^5+\cos\left(\frac{4\pi}{27}\right)^5+\cos\left(\frac{8\pi}{27}\right)^5+\cos\left(\frac{10\pi}{27}\right)^5$$

$$\cos\left(\frac{\pi}{27}\right)^7+\cos\left(\frac{5\pi}{27}\right)^7+\cos\left(\frac{7\pi}{27}\right)^7+\cos\left(\frac{11\pi}{27}\right)^7+\cos\left(\frac{13\pi}{27}\right)^7=\cos\left(\frac{2\pi}{27}\right)^7+\cos\left(\frac{4\pi}{27}\right)^7+\cos\left(\frac{8\pi}{27}\right)^7+\cos\left(\frac{10\pi}{27}\right)^7$$

Example 2.4.4 (Chen Shuwen, July 2021)

$$\sin^k\left(\frac{\pi}{54}\right)+\sin^k\left(\frac{5\pi}{54}\right)+\sin^k\left(\frac{7\pi}{54}\right)+\sin^k\left(\frac{11\pi}{54}\right)+\sin^k\left(\frac{13\pi}{54}\right)+\sin^k\left(\frac{17\pi}{54}\right)+\sin^k\left(\frac{19\pi}{54}\right)+\sin^k\left(\frac{23\pi}{54}\right)+\sin^k\left(\frac{25\pi}{54}\right)$$

$$=\sin^k\left(\frac{2\pi}{54}\right)+\sin^k\left(\frac{4\pi}{54}\right)+\sin^k\left(\frac{8\pi}{54}\right)+\sin^k\left(\frac{10\pi}{54}\right)+\sin^k\left(\frac{14\pi}{54}\right)+\sin^k\left(\frac{16\pi}{54}\right)+\sin^k\left(\frac{20\pi}{54}\right)+\sin^k\left(\frac{22\pi}{54}\right)+\sin^k\left(\frac{26\pi}{54}\right)$$

$$(k=2,4,6,8,10,12,14,16)$$

Example 2.4.5 (Chen Shuwen, 2021/7/4)

$$\left(\cos\frac{0\pi}{11}\right)^k+\left(\cos\frac{2\pi}{11}\right)^k+\left(\cos\frac{2\pi}{11}\right)^k+\left(\cos\frac{4\pi}{11}\right)^k+\left(\cos\frac{4\pi}{11}\right)^k+\left(\cos\frac{6\pi}{11}\right)^k+\left(\cos\frac{6\pi}{11}\right)^k+\left(\cos\frac{8\pi}{11}\right)^k+\left(\cos\frac{8\pi}{11}\right)^k+\left(\cos\frac{10\pi}{11}\right)^k+\left(\cos\frac{10\pi}{11}\right)^k$$

$$=\left(\cos\frac{1\pi}{11}\right)^k+\left(\cos\frac{1\pi}{11}\right)^k+\left(\cos\frac{3\pi}{11}\right)^k+\left(\cos\frac{3\pi}{11}\right)^k+\left(\cos\frac{5\pi}{11}\right)^k+\left(\cos\frac{5\pi}{11}\right)^k+\left(\cos\frac{7\pi}{11}\right)^k+\left(\cos\frac{7\pi}{11}\right)^k+\left(\cos\frac{9\pi}{11}\right)^k+\left(\cos\frac{9\pi}{11}\right)^k+\left(\cos\frac{11\pi}{11}\right)^k$$

$$(k=1,2,3,4,5,6,7,8,9,10)$$

Example 2.4.6 (Chen Shuwen, 2021/7/4)

$$\{a_0,a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10}\}$$

$$=\left\{\cos^2\left(\frac{1\pi}{22}\right),\cos^2\left(\frac{1\pi}{22}\right),\cos^2\left(\frac{3\pi}{22}\right),\cos^2\left(\frac{3\pi}{22}\right),\cos^2\left(\frac{5\pi}{22}\right),\cos^2\left(\frac{5\pi}{22}\right),\cos^2\left(\frac{7\pi}{22}\right),\cos^2\left(\frac{7\pi}{22}\right),\cos^2\left(\frac{9\pi}{22}\right),\cos^2\left(\frac{9\pi}{22}\right),\cos^2\left(\frac{11\pi}{22}\right)\right\}$$

$$\{b_0,b_1,b_2,b_3,b_4,b_5,b_6,b_7,b_8,b_9,b_{10}\}$$

$$=\left\{\cos^2\left(\frac{0\pi}{22}\right),\cos^2\left(\frac{2\pi}{22}\right),\cos^2\left(\frac{2\pi}{22}\right),\cos^2\left(\frac{4\pi}{22}\right),\cos^2\left(\frac{4\pi}{22}\right),\cos^2\left(\frac{6\pi}{22}\right),\cos^2\left(\frac{6\pi}{22}\right),\cos^2\left(\frac{8\pi}{22}\right),\cos^2\left(\frac{8\pi}{22}\right),\cos^2\left(\frac{10\pi}{22}\right),\cos^2\left(\frac{10\pi}{22}\right)\right\}$$

$$a_0^k+a_1^k+a_2^k+a_3^k+a_4^k+a_5^k+a_6^k+a_7^k+a_8^k+a_9^k+a_{10}^k={b_0}^k+{b_1}^k+{b_2}^k+{b_3}^k+{b_4}^k+{b_5}^k+{b_6}^k+{b_7}^k+{b_8}^k+{b_9}^k+{b_{10}}^k$$

$$(k=1,2,3,4,5,6,7,8,9,10)$$

Example 2.4.7 (Chen Shuwen, 2021/7/7)

$$\{a_0,a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10}\}$$

$$=\left\{\sin^2\left(\frac{\pi}{88}\right),\sin^2\left(\frac{7\pi}{88}\right),\sin^2\left(\frac{9\pi}{88}\right),\sin^2\left(\frac{15\pi}{88}\right),\sin^2\left(\frac{17\pi}{88}\right),\sin^2\left(\frac{23\pi}{88}\right),\sin^2\left(\frac{25\pi}{88}\right),\sin^2\left(\frac{31\pi}{88}\right),\sin^2\left(\frac{33\pi}{88}\right),\sin^2\left(\frac{39\pi}{88}\right),\sin^2\left(\frac{41\pi}{88}\right)\right\}$$

$$\{b_0,b_1,b_2,b_3,b_4,b_5,b_6,b_7,b_8,b_9,b_{10}\}$$

$$=\left\{\sin^2\left(\frac{3\pi}{88}\right),\sin^2\left(\frac{5\pi}{88}\right),\sin^2\left(\frac{11\pi}{88}\right),\sin^2\left(\frac{13\pi}{88}\right),\sin^2\left(\frac{19\pi}{88}\right),\sin^2\left(\frac{21\pi}{88}\right),\sin^2\left(\frac{27\pi}{88}\right),\sin^2\left(\frac{29\pi}{88}\right),\sin^2\left(\frac{35\pi}{88}\right),\sin^2\left(\frac{37\pi}{88}\right),\sin^2\left(\frac{43\pi}{88}\right)\right\}$$

$$a_0^k+a_1^k+a_2^k+a_3^k+a_4^k+a_5^k+a_6^k+a_7^k+a_8^k+a_9^k+a_{10}^k$$
$$=b_0^k+b_1^k+b_2^k+b_3^k+b_4^k+b_5^k+b_6^k+b_7^k+b_8^k+b_9^k+b_{10}^k$$
$$(k=1,2,3,4,5,6,7,8,9,10)$$

Example 2.4.8 (Chen Shuwen, July 2021)

$$(\sin\frac{5\pi}{28})^1+(\sin\frac{7\pi}{28})^1+(\sin\frac{13\pi}{28})^1=(\sin\frac{\pi}{28})^1+(\sin\frac{3\pi}{28})^1+(\sin\frac{9\pi}{28})^1+(\sin\frac{11\pi}{28})^1$$

$$(\sin\frac{5\pi}{28})^3+(\sin\frac{7\pi}{28})^3+(\sin\frac{13\pi}{28})^3=(\sin\frac{\pi}{28})^3+(\sin\frac{3\pi}{28})^3+(\sin\frac{9\pi}{28})^3+(\sin\frac{11\pi}{28})^3$$

$$(\sin\frac{5\pi}{28})^5+(\sin\frac{7\pi}{28})^5+(\sin\frac{13\pi}{28})^5=(\sin\frac{\pi}{28})^5+(\sin\frac{3\pi}{28})^5+(\sin\frac{9\pi}{28})^5+(\sin\frac{11\pi}{28})^5$$

$$(\sin\frac{\pi}{32})^2+(\sin\frac{7\pi}{32})^2+(\sin\frac{9\pi}{32})^2+(\sin\frac{15\pi}{32})^2=(\sin\frac{3\pi}{32})^2+(\sin\frac{5\pi}{32})^2+(\sin\frac{11\pi}{32})^2+(\sin\frac{13\pi}{32})^2$$

$$(\sin\frac{\pi}{32})^4+(\sin\frac{7\pi}{32})^4+(\sin\frac{9\pi}{32})^4+(\sin\frac{15\pi}{32})^4=(\sin\frac{3\pi}{32})^4+(\sin\frac{5\pi}{32})^4+(\sin\frac{11\pi}{32})^4+(\sin\frac{13\pi}{32})^4$$

$$(\sin\frac{\pi}{32})^6+(\sin\frac{7\pi}{32})^6+(\sin\frac{9\pi}{32})^6+(\sin\frac{15\pi}{32})^6=(\sin\frac{3\pi}{32})^6+(\sin\frac{5\pi}{32})^6+(\sin\frac{11\pi}{32})^6+(\sin\frac{13\pi}{32})^6$$

$$(\sin\frac{5\pi}{36})^1+(\sin\frac{7\pi}{36})^1+(\sin\frac{13\pi}{36})^1+(\sin\frac{15\pi}{36})^1=(\sin\frac{\pi}{36})^1+(\sin\frac{3\pi}{36})^1+(\sin\frac{9\pi}{36})^1+(\sin\frac{11\pi}{36})^1+(\sin\frac{17\pi}{36})^1$$

$$(\sin\frac{5\pi}{36})^3+(\sin\frac{7\pi}{36})^3+(\sin\frac{13\pi}{36})^3+(\sin\frac{15\pi}{36})^3=(\sin\frac{\pi}{36})^3+(\sin\frac{3\pi}{36})^3+(\sin\frac{9\pi}{36})^3+(\sin\frac{11\pi}{36})^3+(\sin\frac{17\pi}{36})^3$$

$$(\sin\frac{5\pi}{36})^5+(\sin\frac{7\pi}{36})^5+(\sin\frac{13\pi}{36})^5+(\sin\frac{15\pi}{36})^5=(\sin\frac{\pi}{36})^5+(\sin\frac{3\pi}{36})^5+(\sin\frac{9\pi}{36})^5+(\sin\frac{11\pi}{36})^5+(\sin\frac{17\pi}{36})^5$$

$$(\sin\frac{5\pi}{36})^7+(\sin\frac{7\pi}{36})^7+(\sin\frac{13\pi}{36})^7+(\sin\frac{15\pi}{36})^7=(\sin\frac{\pi}{36})^7+(\sin\frac{3\pi}{36})^7+(\sin\frac{9\pi}{36})^7+(\sin\frac{11\pi}{36})^7+(\sin\frac{17\pi}{36})^7$$

Example 2.4.9 (Chen Shuwen, August 2021)

$$\{a_0,a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{12},a_{13},a_{14},a_{15},a_{16},a_{17},a_{18}\}=\{\sin^2\left(\frac{\pi}{1729}\right),\sin^2\left(\frac{90\pi}{1729}\right),\sin^2\left(\frac{92\pi}{1729}\right),$$
$$\sin^2\left(\frac{181\pi}{1729}\right),\sin^2\left(\frac{183\pi}{1729}\right),\sin^2\left(\frac{272\pi}{1729}\right),\sin^2\left(\frac{274\pi}{1729}\right),\sin^2\left(\frac{363\pi}{1729}\right),\sin^2\left(\frac{365\pi}{1729}\right),\sin^2\left(\frac{454\pi}{1729}\right),\sin^2\left(\frac{456\pi}{1729}\right),$$
$$\sin^2\left(\frac{545\pi}{1729}\right),\sin^2\left(\frac{547\pi}{1729}\right),\sin^2\left(\frac{636\pi}{1729}\right),\sin^2\left(\frac{638\pi}{1729}\right),\sin^2\left(\frac{727\pi}{1729}\right),\sin^2\left(\frac{729\pi}{1729}\right),\sin^2\left(\frac{818\pi}{1729}\right),\sin^2\left(\frac{820\pi}{1729}\right)\}$$

$$\{b_0,b_1,b_2,b_3,b_4,b_5,b_6,b_7,b_8,b_9,b_{10},b_{11},b_{12},b_{13},b_{14},b_{15},b_{16},b_{17},b_{18}\}=\{\cos^2\left(\frac{\pi}{1729}\right),\cos^2\left(\frac{90\pi}{1729}\right),\cos^2\left(\frac{92\pi}{1729}\right),$$
$$\cos^2\left(\frac{181\pi}{1729}\right),\cos^2\left(\frac{183\pi}{1729}\right),\cos^2\left(\frac{272\pi}{1729}\right),\cos^2\left(\frac{274\pi}{1729}\right),\cos^2\left(\frac{363\pi}{1729}\right),\cos^2\left(\frac{365\pi}{1729}\right),\cos^2\left(\frac{454\pi}{1729}\right),\cos^2\left(\frac{456\pi}{1729}\right),$$
$$\cos^2\left(\frac{545\pi}{1729}\right),\cos^2\left(\frac{547\pi}{1729}\right),\cos^2\left(\frac{636\pi}{1729}\right),\cos^2\left(\frac{638\pi}{1729}\right),\cos^2\left(\frac{727\pi}{1729}\right),\cos^2\left(\frac{729\pi}{1729}\right),\cos^2\left(\frac{818\pi}{1729}\right),\cos^2\left(\frac{820\pi}{1729}\right)\}$$

$$a_0^k+a_1^k+a_2^k+a_3^k+a_4^k+a_5^k+a_6^k+a_7^k+a_8^k+a_9^k+a_{10}^k+a_{11}^k+a_{12}^k+a_{13}^k+a_{14}^k+a_{15}^k+a_{16}^k+a_{17}^k+a_{18}^k$$
$$=b_0^k+b_1^k+b_2^k+b_3^k+b_4^k+b_5^k+b_6^k+b_7^k+b_8^k+b_9^k+b_{10}^k+b_{11}^k+b_{12}^k+b_{13}^k+b_{14}^k+b_{15}^k+b_{16}^k+b_{17}^k+b_{18}^k$$
$$(k=1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18)$$

Formula 2.5.1 (Chen Shuwen, 2021/8/12)

For positive integer n

$$\frac{1}{2}+\sum_{j=1}^{n-1}\sin^{2k}\left(\frac{\pi j}{2n}\right)=\frac{1}{2}+\sum_{j=1}^{n-1}\cos^{2k}\left(\frac{\pi j}{2n}\right)=\frac{n(2k-1)!}{2^{2k-1}k!\,(k-1)!}=\frac{n\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\,(k+1)}$$

$$(k=1,2,3,\ldots,2n-1)$$

Formula 2.5.2 (Chen Shuwen, 2021/8/12)

For positive odd integer n and any z

$$\sin^{2k}(z)+\sum_{j=1}^{\frac{n-1}{2}}\left(\sin^{2k}\left(\frac{\pi j}{n}-z\right)+\sin^{2k}\left(\frac{\pi j}{n}+z\right)\right)=2\sum_{j=1}^{\frac{n-1}{2}}\sin^{2k}\left(\frac{\pi j}{n}\right)=\frac{n(2k-1)!}{2^{2k-1}k!\,(k-1)!}=\frac{n\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\,(k+1)}$$

$$\cos^{2k}(z)+\sum_{j=1}^{\frac{n-1}{2}}\left(\cos^{2k}\left(\frac{\pi j}{n}-z\right)+\cos^{2k}\left(\frac{\pi j}{n}+z\right)\right)=1+2\sum_{j=1}^{\frac{n-1}{2}}\cos^{2k}\left(\frac{\pi j}{n}\right)=\frac{n(2k-1)!}{2^{2k-1}k!\,(k-1)!}=\frac{n\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\,(k+1)}$$

$$(k=1,2,3,\ldots,n-1)$$

Formula 2.5.3 (Chen Shuwen, 2021/8/12)

For positive odd integer n and any z

$$\cos^{k}(z)+\sum_{j=1}^{\frac{n-1}{2}}\left(\cos^{k}\left(\frac{2\pi j}{n}-z\right)+\cos^{k}\left(\frac{2\pi j}{n}+z\right)\right)=1+2\sum_{j=1}^{\frac{n-1}{2}}\cos^{k}\left(\frac{2\pi j}{n}\right)$$

$$(k=1,2,3,4,\ldots,n-2,n-1,n+1,n+3,n+5,\ldots,2n-2)$$

It is equivalent to

$$\cos^{k}(z)+\sum_{j=1}^{\frac{n-1}{2}}\left(\cos^{k}\left(\frac{2\pi j}{n}-z\right)+\cos^{k}\left(\frac{2\pi j}{n}+z\right)\right)=1+2\sum_{j=1}^{\frac{n-1}{2}}\cos^{k}\left(\frac{2\pi j}{n}\right)=0\qquad(k=1,3,5,\ldots,n-2)$$

$$\cos^{k}(z)+\sum_{j=1}^{\frac{n-1}{2}}\left(\cos^{k}\left(\frac{2\pi j}{n}-z\right)+\cos^{k}\left(\frac{2\pi j}{n}+z\right)\right)=1+2\sum_{j=1}^{\frac{n-1}{2}}\cos^{k}\left(\frac{2\pi j}{n}\right)=\frac{n(k-1)!}{2^{k-1}\left(\frac{k}{2}-1\right)!\,\frac{k}{2}!}=\frac{n\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{k}{2}+1\right)}$$

$$(k=2,4,6,\ldots,2n-2)$$

Formula 2.5.4 (Chen Shuwen, 2021/8/12)

For positive even integer n and any z

$$\sum_{j=1}^{\frac{n}{2}}\left(\sin^{2k}\left(\frac{\pi(2j-1)}{2n}-z\right)+\sin^{2k}\left(\frac{\pi(2j-1)}{2n}+z\right)\right)=2\sum_{j=1}^{\frac{n}{2}}\sin^{2k}\left(\frac{\pi(2j-1)}{2n}\right)=\frac{n(2k-1)!}{2^{2k-1}k!\,(k-1)!}=\frac{n\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\,(k+1)}$$

$$\sum_{j=1}^{\frac{n}{2}}\left(\cos^{2k}\left(\frac{\pi(2j-1)}{2n}-z\right)+\cos^{2k}\left(\frac{\pi(2j-1)}{2n}+z\right)\right)=2\sum_{j=1}^{\frac{n}{2}}\cos^{2k}\left(\frac{\pi(2j-1)}{2n}\right)=\frac{n(2k-1)!}{2^{2k-1}k!\,(k-1)!}=\frac{n\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\left(k+1\right)}$$

$$(k=1,2,3,\ldots,n-1)$$

Example 2.5.1 (Chen Shuwen, August 2021)

$$\cos^{2k}(\frac{0\pi}{100})+\cos^{2k}(\frac{10\pi}{100})+\cos^{2k}(\frac{10\pi}{100})+\cos^{2k}(\frac{20\pi}{100})+\cos^{2k}(\frac{20\pi}{100})+\cos^{2k}(\frac{30\pi}{100})+\cos^{2k}(\frac{30\pi}{100})+\cos^{2k}(\frac{40\pi}{100})+\cos^{2k}(\frac{40\pi}{100})+\cos^{2k}(\frac{50\pi}{100})$$

$$=\cos^{2k}(\frac{1\pi}{100})+\cos^{2k}(\frac{9\pi}{100})+\cos^{2k}(\frac{11\pi}{100})+\cos^{2k}(\frac{19\pi}{100})+\cos^{2k}(\frac{21\pi}{100})+\cos^{2k}(\frac{29\pi}{100})+\cos^{2k}(\frac{31\pi}{100})+\cos^{2k}(\frac{39\pi}{100})+\cos^{2k}(\frac{41\pi}{100})+\cos^{2k}(\frac{49\pi}{100})$$

$$=\cos^{2k}(\frac{2\pi}{100})+\cos^{2k}(\frac{8\pi}{100})+\cos^{2k}(\frac{12\pi}{100})+\cos^{2k}(\frac{18\pi}{100})+\cos^{2k}(\frac{22\pi}{100})+\cos^{2k}(\frac{28\pi}{100})+\cos^{2k}(\frac{32\pi}{100})+\cos^{2k}(\frac{38\pi}{100})+\cos^{2k}(\frac{42\pi}{100})+\cos^{2k}(\frac{48\pi}{100})$$

$$=\cos^{2k}(\frac{3\pi}{100})+\cos^{2k}(\frac{7\pi}{100})+\cos^{2k}(\frac{13\pi}{100})+\cos^{2k}(\frac{17\pi}{100})+\cos^{2k}(\frac{23\pi}{100})+\cos^{2k}(\frac{27\pi}{100})+\cos^{2k}(\frac{33\pi}{100})+\cos^{2k}(\frac{37\pi}{100})+\cos^{2k}(\frac{43\pi}{100})+\cos^{2k}(\frac{47\pi}{100})$$

$$=\cos^{2k}(\frac{4\pi}{100})+\cos^{2k}(\frac{6\pi}{100})+\cos^{2k}(\frac{14\pi}{100})+\cos^{2k}(\frac{16\pi}{100})+\cos^{2k}(\frac{24\pi}{100})+\cos^{2k}(\frac{26\pi}{100})+\cos^{2k}(\frac{34\pi}{100})+\cos^{2k}(\frac{36\pi}{100})+\cos^{2k}(\frac{44\pi}{100})+\cos^{2k}(\frac{46\pi}{100})$$

$$=\cos^{2k}(\frac{5\pi}{100})+\cos^{2k}(\frac{5\pi}{100})+\cos^{2k}(\frac{15\pi}{100})+\cos^{2k}(\frac{15\pi}{100})+\cos^{2k}(\frac{25\pi}{100})+\cos^{2k}(\frac{25\pi}{100})+\cos^{2k}(\frac{35\pi}{100})+\cos^{2k}(\frac{35\pi}{100})+\cos^{2k}(\frac{45\pi}{100})+\cos^{2k}(\frac{45\pi}{100})$$

$$(k=1,2,3,4,5,6,7,8,9)$$

Example 2.5.2 (Chen Shuwen, August 2021)

$$\sin^{2k}(\frac{0\pi}{11})+\sin^{2k}(\frac{1\pi}{11})+\sin^{2k}(\frac{1\pi}{11})+\sin^{2k}(\frac{2\pi}{11})+\sin^{2k}(\frac{2\pi}{11})+\sin^{2k}(\frac{3\pi}{11})+\sin^{2k}(\frac{3\pi}{11})+\sin^{2k}(\frac{4\pi}{11})+\sin^{2k}(\frac{4\pi}{11})+\sin^{2k}(\frac{5\pi}{11})+\sin^{2k}(\frac{5\pi}{11})$$

$$=\sin^{2k}(\frac{0\pi}{22})+\sin^{2k}(\frac{2\pi}{22})+\sin^{2k}(\frac{2\pi}{22})+\sin^{2k}(\frac{4\pi}{22})+\sin^{2k}(\frac{4\pi}{22})+\sin^{2k}(\frac{6\pi}{22})+\sin^{2k}(\frac{6\pi}{22})+\sin^{2k}(\frac{8\pi}{22})+\sin^{2k}(\frac{8\pi}{22})+\sin^{2k}(\frac{10\pi}{22})+\sin^{2k}(\frac{10\pi}{22})$$

$$=\sin^{2k}(\frac{1\pi}{22})+\sin^{2k}(\frac{1\pi}{22})+\sin^{2k}(\frac{3\pi}{22})+\sin^{2k}(\frac{3\pi}{22})+\sin^{2k}(\frac{5\pi}{22})+\sin^{2k}(\frac{5\pi}{22})+\sin^{2k}(\frac{7\pi}{22})+\sin^{2k}(\frac{7\pi}{22})+\sin^{2k}(\frac{9\pi}{22})+\sin^{2k}(\frac{9\pi}{22})+\sin^{2k}(\frac{11\pi}{22})$$

$$=\sin^{2k}(\frac{0\pi}{33})+\sin^{2k}(\frac{3\pi}{33})+\sin^{2k}(\frac{3\pi}{33})+\sin^{2k}(\frac{6\pi}{33})+\sin^{2k}(\frac{6\pi}{33})+\sin^{2k}(\frac{9\pi}{33})+\sin^{2k}(\frac{9\pi}{33})+\sin^{2k}(\frac{12\pi}{33})+\sin^{2k}(\frac{12\pi}{33})+\sin^{2k}(\frac{15\pi}{33})+\sin^{2k}(\frac{15\pi}{33})$$

$$=\sin^{2k}(\frac{1\pi}{33})+\sin^{2k}(\frac{2\pi}{33})+\sin^{2k}(\frac{4\pi}{33})+\sin^{2k}(\frac{5\pi}{33})+\sin^{2k}(\frac{7\pi}{33})+\sin^{2k}(\frac{8\pi}{33})+\sin^{2k}(\frac{10\pi}{33})+\sin^{2k}(\frac{11\pi}{33})+\sin^{2k}(\frac{13\pi}{33})+\sin^{2k}(\frac{14\pi}{33})+\sin^{2k}(\frac{16\pi}{33})$$

$$=\sin^{2k}(\frac{0\pi}{44})+\sin^{2k}(\frac{4\pi}{44})+\sin^{2k}(\frac{4\pi}{44})+\sin^{2k}(\frac{8\pi}{44})+\sin^{2k}(\frac{8\pi}{44})+\sin^{2k}(\frac{12\pi}{44})+\sin^{2k}(\frac{12\pi}{44})+\sin^{2k}(\frac{16\pi}{44})+\sin^{2k}(\frac{16\pi}{44})+\sin^{2k}(\frac{20\pi}{44})+\sin^{2k}(\frac{20\pi}{44})$$

$$=\sin^{2k}(\frac{1\pi}{44})+\sin^{2k}(\frac{3\pi}{44})+\sin^{2k}(\frac{5\pi}{44})+\sin^{2k}(\frac{7\pi}{44})+\sin^{2k}(\frac{9\pi}{44})+\sin^{2k}(\frac{11\pi}{44})+\sin^{2k}(\frac{13\pi}{44})+\sin^{2k}(\frac{15\pi}{44})+\sin^{2k}(\frac{17\pi}{44})+\sin^{2k}(\frac{19\pi}{44})+\sin^{2k}(\frac{21\pi}{44})$$

$$=\sin^{2k}(\frac{2\pi}{44})+\sin^{2k}(\frac{2\pi}{44})+\sin^{2k}(\frac{6\pi}{44})+\sin^{2k}(\frac{6\pi}{44})+\sin^{2k}(\frac{10\pi}{44})+\sin^{2k}(\frac{10\pi}{44})+\sin^{2k}(\frac{14\pi}{44})+\sin^{2k}(\frac{14\pi}{44})+\sin^{2k}(\frac{18\pi}{44})+\sin^{2k}(\frac{18\pi}{44})+\sin^{2k}(\frac{22\pi}{44})$$

$$=\sin^{2k}(\frac{0\pi}{55})+\sin^{2k}(\frac{5\pi}{55})+\sin^{2k}(\frac{5\pi}{55})+\sin^{2k}(\frac{10\pi}{55})+\sin^{2k}(\frac{10\pi}{55})+\sin^{2k}(\frac{15\pi}{55})+\sin^{2k}(\frac{15\pi}{55})+\sin^{2k}(\frac{20\pi}{55})+\sin^{2k}(\frac{20\pi}{55})+\sin^{2k}(\frac{25\pi}{55})+\sin^{2k}(\frac{25\pi}{55})$$

$$=\sin^{2k}(\frac{1\pi}{55})+\sin^{2k}(\frac{4\pi}{55})+\sin^{2k}(\frac{6\pi}{55})+\sin^{2k}(\frac{9\pi}{55})+\sin^{2k}(\frac{11\pi}{55})+\sin^{2k}(\frac{14\pi}{55})+\sin^{2k}(\frac{16\pi}{55})+\sin^{2k}(\frac{19\pi}{55})+\sin^{2k}(\frac{21\pi}{55})+\sin^{2k}(\frac{24\pi}{55})+\sin^{2k}(\frac{26\pi}{55})$$

$$=\sin^{2k}(\frac{2\pi}{55})+\sin^{2k}(\frac{3\pi}{55})+\sin^{2k}(\frac{7\pi}{55})+\sin^{2k}(\frac{8\pi}{55})+\sin^{2k}(\frac{12\pi}{55})+\sin^{2k}(\frac{13\pi}{55})+\sin^{2k}(\frac{17\pi}{55})+\sin^{2k}(\frac{18\pi}{55})+\sin^{2k}(\frac{22\pi}{55})+\sin^{2k}(\frac{23\pi}{55})+\sin^{2k}(\frac{27\pi}{55})$$

$$=\sin^{2k}(\frac{0\pi}{66})+\sin^{2k}(\frac{6\pi}{66})+\sin^{2k}(\frac{6\pi}{66})+\sin^{2k}(\frac{12\pi}{66})+\sin^{2k}(\frac{12\pi}{66})+\sin^{2k}(\frac{18\pi}{66})+\sin^{2k}(\frac{18\pi}{66})+\sin^{2k}(\frac{24\pi}{66})+\sin^{2k}(\frac{24\pi}{66})+\sin^{2k}(\frac{30\pi}{66})+\sin^{2k}(\frac{30\pi}{66})$$

$$= \sin^{2k}(\frac{1\pi}{66}) + \sin^{2k}(\frac{5\pi}{66}) + \sin^{2k}(\frac{7\pi}{66}) + \sin^{2k}(\frac{11\pi}{66}) + \sin^{2k}(\frac{13\pi}{66}) + \sin^{2k}(\frac{17\pi}{66}) + \sin^{2k}(\frac{19\pi}{66}) + \sin^{2k}(\frac{23\pi}{66}) + \sin^{2k}(\frac{25\pi}{66}) + \sin^{2k}(\frac{29\pi}{66}) + \sin^{2k}(\frac{31\pi}{66})$$

$$= \sin^{2k}(\frac{2\pi}{66}) + \sin^{2k}(\frac{4\pi}{66}) + \sin^{2k}(\frac{8\pi}{66}) + \sin^{2k}(\frac{10\pi}{66}) + \sin^{2k}(\frac{14\pi}{66}) + \sin^{2k}(\frac{16\pi}{66}) + \sin^{2k}(\frac{20\pi}{66}) + \sin^{2k}(\frac{22\pi}{66}) + \sin^{2k}(\frac{26\pi}{66}) + \sin^{2k}(\frac{28\pi}{66}) + \sin^{2k}(\frac{32\pi}{66})$$

$$= \sin^{2k}(\frac{3\pi}{66}) + \sin^{2k}(\frac{3\pi}{66}) + \sin^{2k}(\frac{9\pi}{66}) + \sin^{2k}(\frac{9\pi}{66}) + \sin^{2k}(\frac{15\pi}{66}) + \sin^{2k}(\frac{15\pi}{66}) + \sin^{2k}(\frac{21\pi}{66}) + \sin^{2k}(\frac{21\pi}{66}) + \sin^{2k}(\frac{27\pi}{66}) + \sin^{2k}(\frac{27\pi}{66}) + \sin^{2k}(\frac{33\pi}{66})$$

$$= \sin^{2k}(\frac{0\pi}{77}) + \sin^{2k}(\frac{7\pi}{77}) + \sin^{2k}(\frac{7\pi}{77}) + \sin^{2k}(\frac{14\pi}{77}) + \sin^{2k}(\frac{14\pi}{77}) + \sin^{2k}(\frac{21\pi}{77}) + \sin^{2k}(\frac{21\pi}{77}) + \sin^{2k}(\frac{28\pi}{77}) + \sin^{2k}(\frac{28\pi}{77}) + \sin^{2k}(\frac{35\pi}{77}) + \sin^{2k}(\frac{35\pi}{77})$$

$$= \sin^{2k}(\frac{1\pi}{77}) + \sin^{2k}(\frac{6\pi}{77}) + \sin^{2k}(\frac{8\pi}{77}) + \sin^{2k}(\frac{13\pi}{77}) + \sin^{2k}(\frac{15\pi}{77}) + \sin^{2k}(\frac{20\pi}{77}) + \sin^{2k}(\frac{22\pi}{77}) + \sin^{2k}(\frac{27\pi}{77}) + \sin^{2k}(\frac{29\pi}{77}) + \sin^{2k}(\frac{34\pi}{77}) + \sin^{2k}(\frac{36\pi}{77})$$

$$= \sin^{2k}(\frac{2\pi}{77}) + \sin^{2k}(\frac{5\pi}{77}) + \sin^{2k}(\frac{9\pi}{77}) + \sin^{2k}(\frac{12\pi}{77}) + \sin^{2k}(\frac{16\pi}{77}) + \sin^{2k}(\frac{19\pi}{77}) + \sin^{2k}(\frac{23\pi}{77}) + \sin^{2k}(\frac{26\pi}{77}) + \sin^{2k}(\frac{30\pi}{77}) + \sin^{2k}(\frac{33\pi}{77}) + \sin^{2k}(\frac{37\pi}{77})$$

$$= \sin^{2k}(\frac{3\pi}{77}) + \sin^{2k}(\frac{4\pi}{77}) + \sin^{2k}(\frac{10\pi}{77}) + \sin^{2k}(\frac{11\pi}{77}) + \sin^{2k}(\frac{17\pi}{77}) + \sin^{2k}(\frac{18\pi}{77}) + \sin^{2k}(\frac{24\pi}{77}) + \sin^{2k}(\frac{25\pi}{77}) + \sin^{2k}(\frac{31\pi}{77}) + \sin^{2k}(\frac{32\pi}{77}) + \sin^{2k}(\frac{38\pi}{77})$$

$$= \sin^{2k}(\frac{0\pi}{88}) + \sin^{2k}(\frac{8\pi}{88}) + \sin^{2k}(\frac{8\pi}{88}) + \sin^{2k}(\frac{16\pi}{88}) + \sin^{2k}(\frac{16\pi}{88}) + \sin^{2k}(\frac{24\pi}{88}) + \sin^{2k}(\frac{24\pi}{88}) + \sin^{2k}(\frac{32\pi}{88}) + \sin^{2k}(\frac{32\pi}{88}) + \sin^{2k}(\frac{40\pi}{88}) + \sin^{2k}(\frac{40\pi}{88})$$

$$= \sin^{2k}(\frac{1\pi}{88}) + \sin^{2k}(\frac{7\pi}{88}) + \sin^{2k}(\frac{9\pi}{88}) + \sin^{2k}(\frac{15\pi}{88}) + \sin^{2k}(\frac{17\pi}{88}) + \sin^{2k}(\frac{23\pi}{88}) + \sin^{2k}(\frac{25\pi}{88}) + \sin^{2k}(\frac{31\pi}{88}) + \sin^{2k}(\frac{33\pi}{88}) + \sin^{2k}(\frac{39\pi}{88}) + \sin^{2k}(\frac{41\pi}{88})$$

$$= \sin^{2k}(\frac{2\pi}{88}) + \sin^{2k}(\frac{6\pi}{88}) + \sin^{2k}(\frac{10\pi}{88}) + \sin^{2k}(\frac{14\pi}{88}) + \sin^{2k}(\frac{18\pi}{88}) + \sin^{2k}(\frac{22\pi}{88}) + \sin^{2k}(\frac{26\pi}{88}) + \sin^{2k}(\frac{30\pi}{88}) + \sin^{2k}(\frac{34\pi}{88}) + \sin^{2k}(\frac{38\pi}{88}) + \sin^{2k}(\frac{42\pi}{88})$$

$$= \sin^{2k}(\frac{3\pi}{88}) + \sin^{2k}(\frac{5\pi}{88}) + \sin^{2k}(\frac{11\pi}{88}) + \sin^{2k}(\frac{13\pi}{88}) + \sin^{2k}(\frac{19\pi}{88}) + \sin^{2k}(\frac{21\pi}{88}) + \sin^{2k}(\frac{27\pi}{88}) + \sin^{2k}(\frac{29\pi}{88}) + \sin^{2k}(\frac{35\pi}{88}) + \sin^{2k}(\frac{37\pi}{88}) + \sin^{2k}(\frac{43\pi}{88})$$

$$= \sin^{2k}(\frac{4\pi}{88}) + \sin^{2k}(\frac{4\pi}{88}) + \sin^{2k}(\frac{12\pi}{88}) + \sin^{2k}(\frac{12\pi}{88}) + \sin^{2k}(\frac{20\pi}{88}) + \sin^{2k}(\frac{20\pi}{88}) + \sin^{2k}(\frac{28\pi}{88}) + \sin^{2k}(\frac{28\pi}{88}) + \sin^{2k}(\frac{36\pi}{88}) + \sin^{2k}(\frac{36\pi}{88}) + \sin^{2k}(\frac{44\pi}{88})$$

$$= \sin^{2k}(\frac{0\pi}{99}) + \sin^{2k}(\frac{9\pi}{99}) + \sin^{2k}(\frac{9\pi}{99}) + \sin^{2k}(\frac{18\pi}{99}) + \sin^{2k}(\frac{18\pi}{99}) + \sin^{2k}(\frac{27\pi}{99}) + \sin^{2k}(\frac{27\pi}{99}) + \sin^{2k}(\frac{36\pi}{99}) + \sin^{2k}(\frac{36\pi}{99}) + \sin^{2k}(\frac{45\pi}{99}) + \sin^{2k}(\frac{45\pi}{99})$$

$$= \sin^{2k}(\frac{1\pi}{99}) + \sin^{2k}(\frac{8\pi}{99}) + \sin^{2k}(\frac{10\pi}{99}) + \sin^{2k}(\frac{17\pi}{99}) + \sin^{2k}(\frac{19\pi}{99}) + \sin^{2k}(\frac{26\pi}{99}) + \sin^{2k}(\frac{28\pi}{99}) + \sin^{2k}(\frac{35\pi}{99}) + \sin^{2k}(\frac{37\pi}{99}) + \sin^{2k}(\frac{44\pi}{99}) + \sin^{2k}(\frac{46\pi}{99})$$

$$= \sin^{2k}(\frac{2\pi}{99}) + \sin^{2k}(\frac{7\pi}{99}) + \sin^{2k}(\frac{11\pi}{99}) + \sin^{2k}(\frac{16\pi}{99}) + \sin^{2k}(\frac{20\pi}{99}) + \sin^{2k}(\frac{25\pi}{99}) + \sin^{2k}(\frac{29\pi}{99}) + \sin^{2k}(\frac{34\pi}{99}) + \sin^{2k}(\frac{38\pi}{99}) + \sin^{2k}(\frac{43\pi}{99}) + \sin^{2k}(\frac{47\pi}{99})$$

$$= \sin^{2k}(\frac{3\pi}{99}) + \sin^{2k}(\frac{6\pi}{99}) + \sin^{2k}(\frac{12\pi}{99}) + \sin^{2k}(\frac{15\pi}{99}) + \sin^{2k}(\frac{21\pi}{99}) + \sin^{2k}(\frac{24\pi}{99}) + \sin^{2k}(\frac{30\pi}{99}) + \sin^{2k}(\frac{33\pi}{99}) + \sin^{2k}(\frac{39\pi}{99}) + \sin^{2k}(\frac{42\pi}{99}) + \sin^{2k}(\frac{48\pi}{99})$$

$$= \sin^{2k}(\frac{4\pi}{99}) + \sin^{2k}(\frac{5\pi}{99}) + \sin^{2k}(\frac{13\pi}{99}) + \sin^{2k}(\frac{14\pi}{99}) + \sin^{2k}(\frac{22\pi}{99}) + \sin^{2k}(\frac{23\pi}{99}) + \sin^{2k}(\frac{31\pi}{99}) + \sin^{2k}(\frac{32\pi}{99}) + \sin^{2k}(\frac{40\pi}{99}) + \sin^{2k}(\frac{41\pi}{99}) + \sin^{2k}(\frac{49\pi}{99})$$

$$(k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$$

Example 2.5.3 (Chen Shuwen, August 2021)

$$\cos^k\left(\frac{2\pi}{25}\right) + \cos^k\left(\frac{8\pi}{25}\right) + \cos^k\left(\frac{12\pi}{25}\right) + \cos^k\left(\frac{18\pi}{25}\right) + \cos^k\left(\frac{22\pi}{25}\right) = \cos^k\left(\frac{4\pi}{25}\right) + \cos^k\left(\frac{6\pi}{25}\right) + \cos^k\left(\frac{14\pi}{25}\right) + \cos^k\left(\frac{16\pi}{25}\right) + \cos^k\left(\frac{24\pi}{25}\right)$$

$$= \cos^k\left(\frac{\pi}{15}\right) + \cos^k\left(\frac{5\pi}{15}\right) + \cos^k\left(\frac{7\pi}{15}\right) + \cos^k\left(\frac{11\pi}{15}\right) + \cos^k\left(\frac{13\pi}{15}\right)$$

$$= \cos^k\left(\frac{\pi}{10}\right) + \cos^k\left(\frac{3\pi}{10}\right) + \cos^k\left(\frac{5\pi}{10}\right) + \cos^k\left(\frac{7\pi}{10}\right) + \cos^k\left(\frac{9\pi}{10}\right)$$

$$(k = 1, 2, 3, 4, 6, 8)$$

Example 2.5.4 (Chen Shuwen, August 2021)

$$\cos^k\left(\frac{0\pi}{55}\right)+\cos^k\left(\frac{10\pi}{55}\right)+\cos^k\left(\frac{10\pi}{55}\right)+\cos^k\left(\frac{20\pi}{55}\right)+\cos^k\left(\frac{20\pi}{55}\right)+\cos^k\left(\frac{30\pi}{55}\right)+\cos^k\left(\frac{30\pi}{55}\right)+\cos^k\left(\frac{40\pi}{55}\right)+\cos^k\left(\frac{40\pi}{55}\right)+\cos^k\left(\frac{50\pi}{55}\right)+\cos^k\left(\frac{50\pi}{55}\right)$$

$$=\cos^k\left(\frac{1\pi}{55}\right)+\cos^k\left(\frac{9\pi}{55}\right)+\cos^k\left(\frac{11\pi}{55}\right)+\cos^k\left(\frac{19\pi}{55}\right)+\cos^k\left(\frac{21\pi}{55}\right)+\cos^k\left(\frac{29\pi}{55}\right)+\cos^k\left(\frac{31\pi}{55}\right)+\cos^k\left(\frac{39\pi}{55}\right)+\cos^k\left(\frac{41\pi}{55}\right)+\cos^k\left(\frac{49\pi}{55}\right)+\cos^k\left(\frac{51\pi}{55}\right)$$

$$=\cos^k\left(\frac{2\pi}{55}\right)+\cos^k\left(\frac{8\pi}{55}\right)+\cos^k\left(\frac{12\pi}{55}\right)+\cos^k\left(\frac{18\pi}{55}\right)+\cos^k\left(\frac{22\pi}{55}\right)+\cos^k\left(\frac{28\pi}{55}\right)+\cos^k\left(\frac{32\pi}{55}\right)+\cos^k\left(\frac{38\pi}{55}\right)+\cos^k\left(\frac{42\pi}{55}\right)+\cos^k\left(\frac{48\pi}{55}\right)+\cos^k\left(\frac{52\pi}{55}\right)$$

$$=\cos^k\left(\frac{3\pi}{55}\right)+\cos^k\left(\frac{7\pi}{55}\right)+\cos^k\left(\frac{13\pi}{55}\right)+\cos^k\left(\frac{17\pi}{55}\right)+\cos^k\left(\frac{23\pi}{55}\right)+\cos^k\left(\frac{27\pi}{55}\right)+\cos^k\left(\frac{33\pi}{55}\right)+\cos^k\left(\frac{37\pi}{55}\right)+\cos^k\left(\frac{43\pi}{55}\right)+\cos^k\left(\frac{47\pi}{55}\right)+\cos^k\left(\frac{53\pi}{55}\right)$$

$$=\cos^k\left(\frac{4\pi}{55}\right)+\cos^k\left(\frac{6\pi}{55}\right)+\cos^k\left(\frac{14\pi}{55}\right)+\cos^k\left(\frac{16\pi}{55}\right)+\cos^k\left(\frac{24\pi}{55}\right)+\cos^k\left(\frac{26\pi}{55}\right)+\cos^k\left(\frac{34\pi}{55}\right)+\cos^k\left(\frac{36\pi}{55}\right)+\cos^k\left(\frac{44\pi}{55}\right)+\cos^k\left(\frac{46\pi}{55}\right)+\cos^k\left(\frac{54\pi}{55}\right)$$

$$=\cos^k\left(\frac{5\pi}{55}\right)+\cos^k\left(\frac{5\pi}{55}\right)+\cos^k\left(\frac{15\pi}{55}\right)+\cos^k\left(\frac{15\pi}{55}\right)+\cos^k\left(\frac{25\pi}{55}\right)+\cos^k\left(\frac{25\pi}{55}\right)+\cos^k\left(\frac{35\pi}{55}\right)+\cos^k\left(\frac{35\pi}{55}\right)+\cos^k\left(\frac{45\pi}{55}\right)+\cos^k\left(\frac{45\pi}{55}\right)+\cos^k\left(\frac{55\pi}{55}\right)$$

$$=\cos^k\left(\frac{1\pi}{22}\right)+\cos^k\left(\frac{3\pi}{22}\right)+\cos^k\left(\frac{5\pi}{22}\right)+\cos^k\left(\frac{7\pi}{22}\right)+\cos^k\left(\frac{9\pi}{22}\right)+\cos^k\left(\frac{11\pi}{22}\right)+\cos^k\left(\frac{13\pi}{22}\right)+\cos^k\left(\frac{15\pi}{22}\right)+\cos^k\left(\frac{17\pi}{22}\right)+\cos^k\left(\frac{19\pi}{22}\right)+\cos^k\left(\frac{21\pi}{22}\right)$$

$$=\cos^k\left(\frac{1\pi}{33}\right)+\cos^k\left(\frac{5\pi}{33}\right)+\cos^k\left(\frac{7\pi}{33}\right)+\cos^k\left(\frac{11\pi}{33}\right)+\cos^k\left(\frac{13\pi}{33}\right)+\cos^k\left(\frac{17\pi}{33}\right)+\cos^k\left(\frac{19\pi}{33}\right)+\cos^k\left(\frac{23\pi}{33}\right)+\cos^k\left(\frac{25\pi}{33}\right)+\cos^k\left(\frac{29\pi}{33}\right)+\cos^k\left(\frac{31\pi}{33}\right)$$

$$=\cos^k\left(\frac{2\pi}{33}\right)+\cos^k\left(\frac{4\pi}{33}\right)+\cos^k\left(\frac{8\pi}{33}\right)+\cos^k\left(\frac{10\pi}{33}\right)+\cos^k\left(\frac{14\pi}{33}\right)+\cos^k\left(\frac{16\pi}{33}\right)+\cos^k\left(\frac{20\pi}{33}\right)+\cos^k\left(\frac{22\pi}{33}\right)+\cos^k\left(\frac{26\pi}{33}\right)+\cos^k\left(\frac{28\pi}{33}\right)+\cos^k\left(\frac{32\pi}{33}\right)$$

$$=\cos^k\left(\frac{1\pi}{44}\right)+\cos^k\left(\frac{7\pi}{44}\right)+\cos^k\left(\frac{9\pi}{44}\right)+\cos^k\left(\frac{15\pi}{44}\right)+\cos^k\left(\frac{17\pi}{44}\right)+\cos^k\left(\frac{23\pi}{44}\right)+\cos^k\left(\frac{25\pi}{44}\right)+\cos^k\left(\frac{31\pi}{44}\right)+\cos^k\left(\frac{33\pi}{44}\right)+\cos^k\left(\frac{39\pi}{44}\right)+\cos^k\left(\frac{41\pi}{44}\right)$$

$$=\cos^k\left(\frac{3\pi}{44}\right)+\cos^k\left(\frac{5\pi}{44}\right)+\cos^k\left(\frac{11\pi}{44}\right)+\cos^k\left(\frac{13\pi}{44}\right)+\cos^k\left(\frac{19\pi}{44}\right)+\cos^k\left(\frac{21\pi}{44}\right)+\cos^k\left(\frac{27\pi}{44}\right)+\cos^k\left(\frac{29\pi}{44}\right)+\cos^k\left(\frac{35\pi}{44}\right)+\cos^k\left(\frac{37\pi}{44}\right)+\cos^k\left(\frac{43\pi}{44}\right)$$

$$(k=1,2,3,4,5,6,7,8,9,10,12,14,16,18,20)$$

Example 2.5.5 (Chen Shuwen, August 2021)

$$\boldsymbol{a_1=\sqrt{10+2\sqrt{5}}}\qquad \boldsymbol{b_1=\sqrt{8+2\sqrt{10+2\sqrt{5}}}}\qquad \boldsymbol{c_1=\sqrt{8-2\sqrt{8+2\sqrt{10+2\sqrt{5}}}}}\qquad \boldsymbol{d_1=-\sqrt{8-2\sqrt{8-2\sqrt{8+2\sqrt{10+2\sqrt{5}}}}}}$$

$$\boldsymbol{a_2=-\sqrt{10-2\sqrt{5}}}\qquad b_2=\sqrt{8-2\sqrt{10-2\sqrt{5}}}\qquad c_2=\sqrt{8-2\sqrt{8-2\sqrt{10-2\sqrt{5}}}}\qquad d_2=\sqrt{8-2\sqrt{8-2\sqrt{8-2\sqrt{10-2\sqrt{5}}}}}$$

$$\boldsymbol{a_3=0}\qquad b_3=-\sqrt{8}\qquad c_3=-\sqrt{8+4\sqrt{2}}\qquad d_3=\sqrt{8+2\sqrt{8+4\sqrt{2}}}$$

$$\boldsymbol{a_4=\sqrt{10-2\sqrt{5}}}\qquad b_4=-\sqrt{8+2\sqrt{10-2\sqrt{5}}}\qquad c_4=\sqrt{8+2\sqrt{8+2\sqrt{10-2\sqrt{5}}}}\qquad d_4=\sqrt{8-2\sqrt{8+2\sqrt{8+2\sqrt{10-2\sqrt{5}}}}}$$

$$\boldsymbol{a_5=-\sqrt{10+2\sqrt{5}}}\qquad b_5=\sqrt{8-2\sqrt{10+2\sqrt{5}}}\qquad c_5=-\sqrt{8-2\sqrt{8-2\sqrt{10+2\sqrt{5}}}}\qquad d_5=-\sqrt{8+2\sqrt{8-2\sqrt{8-2\sqrt{10+2\sqrt{5}}}}}$$

$$a_1^k+a_2^k+a_3^k+a_4^k+a_5^k=b_1^k+b_2^k+b_3^k+b_4^k+b_5^k=c_1^k+c_2^k+c_3^k+c_4^k+c_5^k=d_1^k+d_2^k+d_3^k+d_4^k+d_5^k$$

$$(k=1,2,3,4,6,8)$$

Chapter 3

The Generalization of Newton Identities

Identities 3.1. Equivalent form of Newton's binomial theorem (Chen Shuwen, 1985)

$$(a+b)^n = \sum_{j=0}^{n} h_j a^j b^{n-j}$$

here

$$h_0 = 1$$

$$h_j = h_{j-1}\frac{(n+1-j)}{j} \qquad (j \geq 1)$$

Example 3.1. Simplification of Pascal's Triangle (Chen Shuwen, 1985, by using Identities 3.1)

For $(a+b)^9$

Step 1

$$[\quad]\ \frac{9}{1}\ [\quad]\ \frac{8}{2}\ [\quad]\ \frac{7}{3}\ [\quad]\ \frac{6}{4}\ [\quad]\ \frac{5}{5}\ [\quad] - [\quad] - [\quad] - [\quad] - [\quad]$$

Step 2

$$[1]\ \frac{9}{1}\ [9]\ \frac{8}{2}\ [36]\ \frac{7}{3}\ [84]\ \frac{6}{4}\ [126]\ \frac{5}{5}\ [\quad] - [\quad] - [\quad] - [\quad] - [\quad]$$

$$(Note:\ 1\times\frac{9}{1}=9,\quad 9\times\frac{8}{2}=36,\quad 36\times\frac{7}{3}=84,\quad 84\times\frac{6}{4}=126\ )$$

Step 3

$$[1]\ \frac{9}{1}\ [9]\ \frac{8}{2}\ [36]\ \frac{7}{3}\ [84]\ \frac{6}{4}\ [126]\ \frac{5}{5}\ [126] - [84] - [36] - [9] - [1]$$

Then we have

$$(a+b)^9 = a^9 + 9a^8b + 36a^7b^2 + 84a^6b^3 + 126a^5b^4 + 126a^4b^5 + 84a^3b^6 + 36a^2b^7 + 9ab^8 + b^9$$

Identities 3.2. A Generalization of Newton's binomial theorem (Chen Shuwen, 1985)

$$(a+b+c)^n = \sum_{i=0}^{n}\sum_{j=0}^{i} h_{i,j} a^j b^{i-j} c^{n-i}$$

here

$$h_{0,0} = 1$$

$$h_{i,0} = \frac{(n+1-i)h_{i-1,0}}{i} \qquad (1 \leq i \leq n)$$

$$h_{i,j} = \frac{(i+1-j)h_{i,j-1}}{j} \qquad (1 \leq j \leq i)$$

Remark. Identities 3.1 and Identities 3.2 can be generalized to $(a_1 + a_2 + \dots + a_m)^n$ for any m.

Example 3.2. The Generalization of Pascal's Triangle (Chen Shuwen, 1985, by using Identities 3.2)
For $(a+b+c)^8$
Step 1. Fill in the bottom row for $(a+b+0)^8$, same as $(a+b)^8$:

$$[1]\ \frac{8}{1}\ [8]\ \frac{7}{2}\ [28]\ \frac{6}{3}\ [56]\ \frac{5}{4}\ [70]\ \frac{4}{5}\ [56]\ \frac{3}{6}\ [28]\ \frac{2}{7}\ [8]\ \frac{1}{8}\ [1]$$

Step 2. Fill in the first number of each row, same data as the bottom row:

[1]

[8] — [ ]

[28] — [ ] — [ ]

[56] — [ ] — [ ] — [ ]

[70] — [ ] — [ ] — [ ] — [ ]

[56] — [ ] — [ ] — [ ] — [ ] — [ ]

[28] — [ ] — [ ] — [ ] — [ ] — [ ] — [ ]

[8] — [ ] — [ ] — [ ] — [ ] — [ ] — [ ] — [ ]

$$[1]\ \frac{8}{1}\ [8]\ \frac{7}{2}\ [28]\ \frac{6}{3}\ [56]\ \frac{5}{4}\ [70]\ \frac{4}{5}\ [56]\ \frac{3}{6}\ [28]\ \frac{2}{7}\ [8]\ \frac{1}{8}\ [1]$$

Step 3. Fill in the rest numbers, same method as the bottom row:

$$[1]$$

$$[8]\ \frac{1}{1}\ [8]$$

$$[28]\ \frac{2}{1}\ [56]\ \frac{1}{2}\ [28]$$

$$[56]\ \frac{3}{1}\ [168]\ \frac{2}{2}\ [168]\ \frac{1}{3}\ [56]$$

$$[70]\ \frac{4}{1}\ [280]\ \frac{3}{2}\ [420]\ \frac{2}{3}\ [280]\ \frac{1}{4}\ [70]$$

$$[56]\ \frac{5}{1}\ [280]\ \frac{4}{2}\ [560]\ \frac{3}{3}\ [560]\ \frac{2}{4}\ [280]\ \frac{1}{5}\ [56]$$

$$[28]\ \frac{6}{1}\ [168]\ \frac{5}{2}\ [420]\ \frac{4}{3}\ [560]\ \frac{3}{4}\ [420]\ \frac{2}{5}\ [168]\ \frac{1}{6}\ [28]$$

$$[8]\ \frac{7}{1}\ [56]\ \frac{6}{2}\ [168]\ \frac{5}{3}\ [280]\ \frac{4}{4}\ [280]\ \frac{3}{5}\ [168]\ \frac{2}{6}\ [56]\ \frac{1}{7}\ [8]$$

$$[1]\ \frac{8}{1}\ [8]\ \frac{7}{2}\ [28]\ \frac{6}{3}\ [56]\ \frac{5}{4}\ [70]\ \frac{4}{5}\ [56]\ \frac{3}{6}\ [28]\ \frac{2}{7}\ [8]\ \frac{1}{8}\ [1]$$

Then we have

$$(a+b+c)^8 = c^8$$
$$+8ac^7 + 8bc^7$$

$$+28a^2c^6 + 56abc^6 + 28b^2c^6$$
$$+56a^3c^5 + 168a^2bc^5 + 168ab^2c^5 + 56b^3c^5$$
$$+70a^4c^4 + 280a^3bc^4 + 420a^2b^2c^4 + 280ab^3c^4 + 70b^4c^4$$
$$+56a^5c^3 + 280a^4bc^3 + 560a^3b^2c^3 + 560a^2b^3c^3 + 280ab^4c^3 + 56b^5c^3$$
$$+28a^6c^2 + 168a^5bc^2 + 420a^4b^2c^2 + 560a^3b^3c^2 + 420a^2b^4c^2 + 168ab^5c^2 + 28b^6c^2$$
$$+8a^7c + 56a^6bc + 168a^5b^2c + 280a^4b^3c + 280a^3b^4c + 168a^2b^5c + 56ab^6c + 8b^7c$$
$$+a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$$

Identities 3.3. (The Girard-Newton Identities, Classical form, by Girard and Newton)
Let n and k be positive integer. Denote

$$P_k = a_1^k + a_2^k + a_3^k + \cdots + a_n^k$$

and

$$S_1 = -(a_1 + a_2 + a_3 + \cdots + a_n)$$
$$S_2 = a_1a_2 + a_1a_3 + a_2a_3 + \cdots + a_{n-1}a_n$$
$$S_3 = -(a_1a_2a_3 + \cdots + a_{n-2}a_{n-1}a_n)$$
$$\vdots$$
$$S_n = (-1)^n(a_1a_2a_3 \dots a_n)$$

then

$$P_1 + 1S_1 = 0$$
$$P_2 + S_1P_1 + 2S_2 = 0$$
$$P_3 + S_1P_2 + S_2P_1 + 3S_3 = 0$$
$$\vdots$$
$$P_n + S_1P_{n-1} + S_2P_{n-2} + \cdots + S_{n-1}P_1 + nS_n = 0$$
$$P_k + S_1P_{k-1} + S_2P_{k-2} + \cdots + S_{n-1}P_{k-n+1} + S_nP_{k-n} = 0 \qquad \text{for } k > n$$

Identities 3.4. (Equivalent form of The Girard-Newton Identities, by Chen Shuwen, 2017, 2019)
Let n and k be positive integer. Denote

$$P_k = a_1^k + a_2^k + a_3^k + \cdots a_n^k$$

and denote

$$S_1 = -(P_1)/1$$
$$S_2 = -(P_2 + S_1P_1)/2$$
$$S_3 = -(P_3 + S_1P_2 + S_2P_1)/3$$
$$\vdots$$
$$S_k = -(P_k + S_1P_{k-1} + S_2P_{k-2} + \cdots + S_{k-1}P_1)/k$$

Then

$$S_1 = -(a_1 + a_2 + a_3 + \cdots + a_n)$$
$$S_2 = a_1a_2 + a_1a_3 + a_2a_3 + \cdots + a_{n-1}a_n$$
$$S_3 = -(a_1a_2a_3 + \cdots + a_{n-2}a_{n-1}a_n)$$
$$\vdots$$
$$S_k = (-1)^k \sum_{1 \le j_1 < j_2 < \cdots < j_k \le n} a_{j_1}a_{j_2} \dots a_{j_k}$$
$$\vdots$$
$$S_n = (-1)^n(a_1a_2a_3 \dots a_n)$$
$$S_k = 0 \qquad \text{for } k > n$$

and

$$P_1 + 1S_1 = 0$$
$$P_2 + S_1P_1 + 2S_2 = 0$$
$$P_3 + S_1P_2 + S_2P_1 + 3S_3 = 0$$
$$\vdots$$
$$P_n + S_1P_{n-1} + S_2P_{n-2} + \cdots + S_{n-1}P_1 + nS_n = 0$$
$$P_k + S_1P_{k-1} + S_2P_{k-2} + \cdots + S_{n-1}P_{k-n+1} + S_nP_{k-n} = 0 \qquad \text{for } k > n$$

Example 3.3. ($n = 5$ case of Identities 3.4)

Let k be positive integer. Denote

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k$$

and Denote

$$S_1 = -(P_1)/1$$
$$S_2 = -(P_2 + S_1P_1)/2$$
$$S_3 = -(P_3 + S_1P_2 + S_2P_1)/3$$
$$S_4 = -(P_4 + S_1P_3 + S_2P_2 + S_3P_1)/4$$
$$S_5 = -(P_5 + S_1P_4 + S_2P_3 + S_3P_2 + S_4P_1)/5$$
$$S_6 = -(P_6 + S_1P_5 + S_2P_4 + S_3P_3 + S_4P_2 + S_5P_1)/6$$
$$S_7 = -(P_7 + S_1P_6 + S_2P_5 + S_3P_4 + S_4P_3 + S_5P_2 + S_6P_1)/7$$
$$S_8 = -(P_8 + S_1P_7 + S_2P_6 + S_3P_5 + S_4P_4 + S_5P_3 + S_6P_2 + S_7P_1)/8$$
$$\vdots$$
$$S_k = -(P_k + S_1P_{k-1} + S_2P_{k-2} + \cdots + S_{k-1}P_1)/k$$

Then

$$S_1 = -(a_1 + a_2 + a_3 + a_4 + a_5)$$
$$S_2 = a_1a_2 + a_1a_3 + a_2a_3 + \cdots + a_4a_5$$
$$S_3 = -(a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + \cdots + a_3a_4a_5)$$
$$S_4 = a_1a_2a_3a_4 + a_1a_2a_3a_5 + a_1a_2a_4a_5 + a_1a_3a_4a_5 + a_2a_3a_4a_5$$
$$S_5 = -a_1a_2a_3a_4a_5$$
$$S_6 = 0$$
$$S_7 = 0$$
$$S_8 = 0$$
$$\vdots$$
$$S_k = 0 \qquad \text{for } k > 5$$

and

$$P_1 + 1S_1 = 0$$
$$P_2 + S_1P_1 + 2S_2 = 0$$
$$P_3 + S_1P_2 + S_2P_1 + 3S_3 = 0$$
$$P_4 + S_1P_3 + S_2P_2 + S_3P_1 + 4S_4 = 0$$
$$P_5 + S_1P_4 + S_2P_3 + S_3P_2 + S_4P_1 + 5S_5 = 0$$
$$P_6 + S_1P_5 + S_2P_4 + S_3P_3 + S_4P_2 + S_5P_1 = 0$$
$$P_7 + S_1P_6 + S_2P_5 + S_3P_4 + S_4P_3 + S_5P_2 = 0$$
$$P_8 + S_1P_7 + S_2P_6 + S_3P_5 + S_4P_4 + S_5P_3 = 0$$
$$\vdots$$
$$P_k + S_1P_{k-1} + S_2P_{k-2} + S_3P_{k-3} + S_4P_{k-4} + S_5P_{k-5} = 0 \qquad \text{for } k > 5$$

Identities 3.5. (Chen Shuwen, 2017, 2019)

Let

$$S_1 = -|\,P_1\,| \,/\, 1!$$

$$S_2 = \begin{vmatrix} P_1 & 1 \\ P_2 & P_1 \end{vmatrix} \,/\, 2!$$

$$S_3 = -\begin{vmatrix} P_1 & 1 & 0 \\ P_2 & P_1 & 2 \\ P_3 & P_2 & P_1 \end{vmatrix} \,/\, 3!$$

$$S_4 = \begin{vmatrix} P_1 & 1 & 0 & 0 \\ P_2 & P_1 & 2 & 0 \\ P_3 & P_2 & P_1 & 3 \\ P_4 & P_3 & P_2 & P_1 \end{vmatrix} \,/\, 4!$$

$$S_5 = -\begin{vmatrix} P_1 & 1 & 0 & 0 & 0 \\ P_2 & P_1 & 2 & 0 & 0 \\ P_3 & P_2 & P_1 & 3 & 0 \\ P_4 & P_3 & P_2 & P_1 & 4 \\ P_5 & P_4 & P_3 & P_2 & P_1 \end{vmatrix} \,/\, 5!$$

$$S_6 = \begin{vmatrix} P_1 & 1 & 0 & 0 & 0 & 0 \\ P_2 & P_1 & 2 & 0 & 0 & 0 \\ P_3 & P_2 & P_1 & 3 & 0 & 0 \\ P_4 & P_3 & P_2 & P_1 & 4 & 0 \\ P_5 & P_4 & P_3 & P_2 & P_1 & 5 \\ P_6 & P_5 & P_4 & P_3 & P_2 & P_1 \end{vmatrix} \,/\, 6!$$

$$S_7 = -\begin{vmatrix} P_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ P_2 & P_1 & 2 & 0 & 0 & 0 & 0 \\ P_3 & P_2 & P_1 & 3 & 0 & 0 & 0 \\ P_4 & P_3 & P_2 & P_1 & 4 & 0 & 0 \\ P_5 & P_4 & P_3 & P_2 & P_1 & 5 & 0 \\ P_6 & P_5 & P_4 & P_3 & P_2 & P_1 & 6 \\ P_7 & P_6 & P_5 & P_4 & P_3 & P_2 & P_1 \end{vmatrix} \,/\, 7!$$

$$S_8 = \begin{vmatrix} P_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ P_2 & P_1 & 2 & 0 & 0 & 0 & 0 & 0 \\ P_3 & P_2 & P_1 & 3 & 0 & 0 & 0 & 0 \\ P_4 & P_3 & P_2 & P_1 & 4 & 0 & 0 & 0 \\ P_5 & P_4 & P_3 & P_2 & P_1 & 5 & 0 & 0 \\ P_6 & P_5 & P_4 & P_3 & P_2 & P_1 & 6 & 0 \\ P_7 & P_6 & P_5 & P_4 & P_3 & P_2 & P_1 & 7 \\ P_8 & P_7 & P_6 & P_5 & P_4 & P_3 & P_2 & P_1 \end{vmatrix} \,/\, 8!$$

$$S_k = (-1)^k \begin{vmatrix} P_1 & 1 & 0 & \dots & 0 & 0 & 0 \\ P_2 & P_1 & 2 & \dots & 0 & 0 & 0 \\ P_3 & P_2 & P_1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ P_{k-2} & P_{k-3} & P_{k-4} & \dots & P_1 & k-2 & 0 \\ P_{k-1} & P_{k-2} & P_{k-3} & \dots & P_2 & P_1 & k-1 \\ P_k & P_{k-1} & P_{k-2} & \dots & P_3 & P_2 & P_1 \end{vmatrix} \,/\, k!$$

then the above relations are equivalent to

$$S_1 = -(P_1)/1$$
$$S_2 = -(P_2 + S_1P_1)/2$$
$$S_3 = -(P_3 + S_1P_2 + S_2P_1)/3$$
$$S_4 = -(P_4 + S_1P_3 + S_2P_2 + S_3P_1)/4$$
$$S_5 = -(P_5 + S_1P_4 + S_2P_3 + S_3P_2 + S_4P_1)/5$$
$$S_6 = -(P_6 + S_1P_5 + S_2P_4 + S_3P_3 + S_4P_2 + S_5P_1)/6$$
$$S_7 = -(P_7 + S_1P_6 + S_2P_5 + S_3P_4 + S_4P_3 + S_5P_2 + S_6P_1)/7$$
$$S_8 = -(P_8 + S_1P_7 + S_2P_6 + S_3P_5 + S_4P_4 + S_5P_3 + S_6P_2 + S_7P_1)/8$$
$$\vdots$$
$$S_k = -(P_k + S_1P_{k-1} + S_2P_{k-2} + \cdots + S_{k-1}P_1)/k$$

Definition 3.1. (P form of S_k)

By expand the below relations of S_k and P_k

$$S_1 = -(P_1)/1$$
$$S_2 = -(P_2 + S_1P_1)/2$$
$$S_3 = -(P_3 + S_1P_2 + S_2P_1)/3$$
$$\vdots$$
$$S_k = -(P_k + S_1P_{k-1} + S_2P_{k-2} + \cdots + S_{k-1}P_1)/k$$

We have

$$S_1 = -P_1$$

$$S_2 = \frac{P_1^2}{2} - \frac{P_2}{2}$$

$$S_3 = -\frac{P_1^3}{6} + \frac{P_1P_2}{2} - \frac{P_3}{3}$$

$$S_4 = \frac{P_1^4}{24} - \frac{1}{4}P_1^2P_2 + \frac{P_2^2}{8} + \frac{P_1P_3}{3} - \frac{P_4}{4}$$

$$S_5 = -\frac{P_1^5}{120} + \frac{1}{12}P_1^3P_2 - \frac{1}{8}P_1P_2^2 - \frac{1}{6}P_1^2P_3 + \frac{P_2P_3}{6} + \frac{P_1P_4}{4} - \frac{P_5}{5}$$

$$S_6 = \frac{P_1^6}{720} - \frac{1}{48}P_1^4P_2 + \frac{1}{16}P_1^2P_2^2 - \frac{P_2^3}{48} + \frac{1}{18}P_1^3P_3 - \frac{1}{6}P_1P_2P_3 + \frac{P_3^2}{18} - \frac{1}{8}P_1^2P_4 + \frac{P_2P_4}{8} + \frac{P_1P_5}{5} - \frac{P_6}{6}$$

$$S_7 = -\frac{P_1^7}{5040} + \frac{1}{240}P_1^5P_2 - \frac{1}{48}P_1^3P_2^2 + \frac{1}{48}P_1P_2^3 - \frac{1}{72}P_1^4P_3 + \frac{1}{12}P_1^2P_2P_3 - \frac{1}{24}P_2^2P_3 - \frac{1}{18}P_1P_3^2 + \frac{1}{24}P_1^3P_4$$
$$- \frac{1}{8}P_1P_2P_4 + \frac{P_3P_4}{12} - \frac{1}{10}P_1^2P_5 + \frac{P_2P_5}{10} + \frac{P_1P_6}{6} - \frac{P_7}{7}$$

… …

Generally

$$S_k = \sum_{\substack{0\le i_1,i_2,i_3,\ldots,i_k\le k \\ i_1+2i_2+\cdots+ki_k=k}} \frac{(-P_1/1)^{i_1}(-P_2/2)^{i_2}\ldots(-P_k/k)^{i_k}}{i_1!\,i_2!\ldots i_k!}$$

We call it the P form of S_k.

Definition 3.2. (Q form of S_k , Chen Shuwen, 2017)

For the relation of S_k and P_k

$$S_1 = -(P_1)/1$$

$$S_2 = -(P_2 + S_1P_1)/2$$

$$S_3 = -(P_3 + S_1P_2 + S_2P_1)/3$$

$$\vdots$$

$$S_k = -(P_k + S_1P_{k-1} + S_2P_{k-2} + \cdots + S_{k-1}P_1)/k$$

If we let

$$P_k = -k\,Q_k$$

then

$$S_1 = Q_1$$

$$S_2 = \frac{Q_1^2}{2!} + Q_2$$

$$S_3 = \frac{Q_1^3}{3!} + Q_1Q_2 + Q_3$$

$$S_4 = \frac{Q_1^4}{4!} + \frac{Q_1^2}{2!}Q_2 + \frac{Q_2^2}{2!} + Q_1Q_3 + Q_4$$

$$S_5 = \frac{Q_1^5}{5!} + \frac{Q_1^3}{3!}Q_2 + \frac{Q_2^2}{2!}Q_1 + \frac{Q_1^2}{2!}Q_3 + Q_2Q_3 + Q_1Q_4 + Q_5$$

$$S_6 = \frac{Q_1^6}{6!} + \frac{Q_1^4}{4!}Q_2 + \frac{Q_2^2}{2!}\frac{Q_1^2}{2!} + \frac{{Q_2}^3}{3!} + \frac{Q_1^3}{3!}Q_3 + Q_1Q_2Q_3 + \frac{Q_3^2}{2!} + \frac{Q_1^2}{2!}Q_4 + Q_2Q_4 + Q_1Q_5 + Q_6$$

$$S_7 = \frac{Q_1^7}{7!} + \frac{Q_1^5}{5!}Q_2 + \frac{Q_1^3}{3!}\frac{Q_2^2}{2!} + Q_1\frac{{Q_2}^3}{3!} + \frac{Q_1^4}{4!}Q_3 + \frac{Q_1^2}{2!}Q_2Q_3 + \frac{{Q_2}^2}{2!}Q_3 + Q_1\frac{Q_3^2}{2!} + \frac{Q_1^3}{3!}Q_4 + Q_1Q_2Q_4 + Q_3Q_4 + \frac{Q_1^2}{2!}Q_5 + Q_2Q_5 + Q_1Q_6 + Q_7$$

… …

Generally

$$S_k = \sum_{\substack{0\le i_1,i_2,i_3,\dots,i_k\le k \\ i_1+2i_2+\cdots+k\ _k=k}} \frac{{Q_1}^{i_1}Q_2^{i_2}\dots Q_k^{i_k}}{i_1!\,i_2!\dots i_k!}$$

We call it the Q form of S_k. In case of Q form, the sign of each item is always positive.

Definition 3.3. (F form of S_k, Chen Shuwen, 1999)

For the relation of S_k and P_k

$$S_1 = -(P_1)/1$$
$$S_2 = -(P_2 + S_1P_1)/2$$
$$S_3 = -(P_3 + S_1P_2 + S_2P_1)/3$$
$$\vdots$$
$$S_k = -(P_k + S_1P_{k-1} + S_2P_{k-2} + \cdots + S_{k-1}P_1)/k$$

If we let

$$P_1 = -F_1$$
$$P_k = P_1^k - kF_k \qquad \text{for } k > 1$$

then

$$S_1 = F_1$$
$$S_2 = F_2$$
$$S_3 = F_1F_2 + F_3$$
$$S_4 = \frac{F_2^2}{2} + F_1F_3 + F_4$$
$$S_5 = \frac{1}{2}F_1F_2^2 + F_2F_3 + F_1F_4 + F_5$$
$$S_6 = \frac{F_2^3}{6} + F_1F_2F_3 + \frac{F_3^2}{2} + F_2F_4 + F_1F_5 + F_6$$
$$S_7 = \frac{1}{6}F_1F_2^3 + \frac{1}{2}F_2^2F_3 + \frac{1}{2}F_1F_3^2 + F_1F_2F_4 + F_3F_4 + F_2F_5 + F_1F_6 + F_7$$
$$S_8 = \frac{F_2^4}{24} + \frac{1}{2}F_1F_2^2F_3 + \frac{1}{2}F_2F_3^2 + \frac{1}{2}F_2^2F_4 + F_1F_3F_4 + \frac{F_4^2}{2} + F_1F_2F_5 + F_3F_5 + F_2F_6 + F_1F_7 + F_8$$

… …

Generally

$$S_k = \sum_{\substack{0\le i_1\le 1 \\ 0\le i_2,i_3,\dots,i_k\le k \\ i_1+2i_2+\cdots+k\ _k=k}} \frac{{F_1}^{i_1}F_2^{i_2}F_3^{i_3}\dots F_k^{i_k}}{i_2!\,i_3!\dots i_k!}$$

We call it the F form of S_k. In case of F form, the sign of each item is positive, and the power exponent of F_1 is always 1.

Definition 3.4. (D form of S_k , Chen Shuwen, 2020)
For the relation of S_k and P_k

$$S_1 = -(P_1)/1$$
$$S_2 = -(P_2 + S_1P_1)/2$$
$$S_3 = -(P_3 + S_1P_2 + S_2P_1)/3$$
$$\vdots$$
$$S_k = -(P_k + S_1P_{k-1} + S_2P_{k-2} + \cdots + S_{k-1}P_1)/k$$

If we let

$$P_1 = -D_1$$
$$P_2 = D_1^2 - 2D_2$$
$$P_3 = -D_1^3 - 3D_3$$
$$P_4 = D_1^4 + 2D_2^2 - 4D_4 \square$$
$$P_5 = -D_1^5 - 5D_5$$
$$P_6 = D_1^6 - 2D_2^3 + 3D_3^2 - 6D_6$$
$$P_7 = -D_1^7 - 7D_7$$
$$P_8 = D_1^8 + 2D_2^4 + 4D_4^2 - 8D_8$$
$$P_9 = -D_1^9 - 3D_3^3 - 9D_9$$
$$P_{10} = D_1^{10} - 2D_2^5 + 5D_5^2 - 10D_{10}$$
$$P_{11} = -D_1^{11} - 11D_{11}$$
$$\vdots$$
$$P_k = \sum_{j=1}^{h} (-1)^{\left(\frac{k}{f_j}\right)} f_j D_{f_j}^{\left(\frac{k}{f_j}\right)}$$

here $f_1, f_2, \ldots, f_h$ are all positive factors of k, include 1 and k itself.
then

$$S_1 = D_1$$
$$S_2 = D_2$$
$$S_3 = D_1D_2 + D_3$$
$$S_4 = D_1D_3 + D_4$$
$$S_5 = D_2D_3 + D_1D_4 + D_5$$
$$S_6 = D_1D_2D_3 + D_2D_4 + D_1D_5 + D_6$$
$$S_7 = D_1D_2D_4 + D_3D_4 + D_2D_5 + D_1D_6 + D_7$$
$$S_8 = D_1D_3D_4 + D_1D_2D_5 + D_3D_5 + D_2D_6 + D_1D_7 + D_8$$
$$S_9 = D_2D_3D_4 + D_1D_3D_5 + D_4D_5 + D_1D_2D_6 + D_3D_6 + D_2D_7 + D_1D_8 + D_9$$
$$S_{10} = D_1D_2D_3D_4 + D_2D_3D_5 + D_1D_4D_5 + D_1D_3D_6 + D_4D_6 + D_1D_2D_7 + D_3D_7 + D_2D_8 + D_1D_9 + D_{10}$$
$$S_{11} = D_1D_2D_3D_5 + D_2D_4D_5 + D_2D_3D_6 + D_1D_4D_6 + D_5D_6 + D_1D_3D_7 + D_4D_7 + D_1D_2D_8 + D_3D_8 + D_2D_9 + D_1D_{10} + D_{11}$$
$$\ldots \ldots$$

Generally

$$S_k = \sum_{\substack{0 \le i_1, i_2, i_3, \ldots, i_k \le 1 \\ i_1 + 2i_2 + \cdots + k\ _k = k}} D_1^{i_1} D_2^{i_2} D_3^{i_3} \ldots D_k^{i_k}$$

We call it the D form of S_k. In case of D form, the sign of each item is positive, the coefficient of each item is always 1, and the power exponents of $D_1, D_2, \ldots, D_k$ all are 1.

Example 3.3.

Comparation of P form, Q form, F form, and D form:

$$S_7=-\frac{P_1^7}{5040}+\frac{1}{240}P_1^5P_2-\frac{1}{48}P_1^3P_2^2+\frac{1}{48}P_1P_2^3-\frac{1}{72}P_1^4P_3+\frac{1}{12}P_1^2P_2P_3-\frac{1}{24}P_2^2P_3-\frac{1}{18}P_1P_3^2+\frac{1}{24}P_1^3P_4$$
$$-\frac{1}{8}P_1P_2P_4+\frac{P_3P_4}{12}-\frac{1}{10}P_1^2P_5+\frac{P_2P_5}{10}+\frac{P_1P_6}{6}-\frac{P_7}{7}$$

$$S_7=\frac{Q_1^7}{7!}+\frac{Q_1^5}{5!}Q_2+\frac{Q_1^3}{3!}\frac{Q_2^2}{2!}+Q_1\frac{{Q_2}^3}{3!}+\frac{Q_1^4}{4!}Q_3+\frac{Q_1^2}{2!}Q_2Q_3+\frac{{Q_2}^2}{2!}Q_3+Q_1\frac{Q_3^2}{2!}+\frac{Q_1^3}{3!}Q_4+Q_1Q_2Q_4+Q_3Q_4+\frac{Q_1^2}{2!}Q_5$$
$$+Q_2Q_5+Q_1Q_6+Q_7$$

$$S_7=\frac{F_2^3}{3!}F_1+\frac{F_2^2}{2!}F_3+\frac{F_3^2}{2!}F_1+F_1F_2F_4+F_3F_4+F_2F_5+F_1F_6+F_7$$

$$S_7=D_1D_2D_4+D_3D_4+D_2D_5+D_1D_6+D_7$$

Definition 3.4.1. An equivalent definition of *D* form

The below relation in Definition 3.4.

$$P_k=\sum_{j=1}^{h}(-1)^{\left(\frac{k}{f_j}\right)}f_jD_{f_j}^{\left(\frac{k}{f_j}\right)}$$

here $f_1, f_2, \dots, f_h$ are all positive factors of k, include 1 and k itself.

is equivalent to

$P_1=-D_1$

$P_k=(-1)^kD_1^k-kD_k$ — k is prime number

$P_k=(-1)^kD_1^k-kD_k+\sum_{j=1}^{h}(-1)^{\left(\frac{k}{f_j}\right)}f_jD_{f_j}^{\left(\frac{k}{f_j}\right)}$ — k is composite number with all its positive factors $f_1, f_2, \dots, f_h$, exclude 1 and k itself.

Example:

$P_1=-D_1$
$P_2=D_1^2-2D_2$
$P_3=-D_1^3-3D_3$
$P_4=D_1^4-4D_4+(2D_2^2)$□
$P_5=-D_1^5-5D_5$
$P_6=D_1^6-6D_6+(-2D_2^3+3D_3^2)$
$P_7=-D_1^7-7D_7$
$P_8=D_1^8-8D_8+(2D_2^4+4D_4^2)$
$P_9=-D_1^9-9D_9+(-3D_3^3)$
$P_{10}=D_1^{10}-10D_{10}+(-2D_2^5+5D_5^2)$
$P_{11}=-D_1^{11}-11D_{11}$
$P_{12}=D_1^{12}-12D_{12}+(2D_2^6+3D_3^4-4D_4^3+6D_6^2)$
$P_{13}=-D_1^{13}-13D_{13}$
$P_{14}=D_1^{14}-14D_{14}+(-2D_2^7+7D_7^2)$
$P_{15}=-D_1^{15}-15D_{15}+(-3D_3^5-5D_5^3)$
$P_{16}=D_1^{16}-16D_{16}+(2D_2^8+4D_4^4+8D_8^2)$
$P_{17}=-D_1^{17}-17D_{17}$
……

It is very interesting that the relation of P_k and D_k shows the primeness and factors of k.

Identities 3.6. (P form of The Girard-Newton Identities, by Chen Shuwen, 2017-2019)
Let n and k be positive integer. Denote

$$P_k = a_1^k + a_2^k + a_3^k + \cdots a_n^k$$

and

$$S_k = \sum_{\substack{0 \le i_1, i_2, i_3, \dots, i_k \le k \\ i_1 + 2i_2 + \cdots + ki_k = k}} \frac{(-P_1/1)^{i_1}(-P_2/2)^{i_2} \dots (-P_k/k)^{i_k}}{i_1!\, i_2! \dots i_k!}$$

then

$$S_k = (-1)^k \sum_{1 \le j_1 < j_2 < \cdots < j_k \le n} a_{j_1} a_{j_2} \dots a_{j_k} \qquad k \le n$$

$$S_k = 0 \qquad k > n$$

Identities 3.7. (Q form of The Girard-Newton Identities, by Chen Shuwen, 2017-2019)
Let n and k be positive integers. Denote

$$P_k = a_1^k + a_2^k + \cdots + a_n^k$$

and

$$Q_k = -P_k/k$$

and

$$S_k = \sum_{\substack{0 \le i_1, i_2, i_3, \dots, i_k \le k \\ i_1 + 2i_2 + \cdots + ki_k = k}} \frac{Q_1{}^{i_1} Q_2^{i_2} \dots Q_k^{i_k}}{i_1!\, i_2! \dots i_k!}$$

then

$$S_k = (-1)^k \sum_{1 \le j_1 < j_2 < \cdots < j_k \le n} a_{j_1} a_{j_2} \dots a_{j_k} \qquad k \le n$$

$$S_k = 0 \qquad k > n$$

Identities 3.8. (F form of The Girard-Newton Identities, by Chen Shuwen, 2017-2019)
Let n and k be positive integers. Denote

$$P_k = a_1^k + a_2^k + \cdots + a_n^k$$

and

$$F_1 = -P_1$$

$$F_k = \left(P_1^k - P_k\right)/k \qquad k > 1$$

and

$$S_k = \sum_{\substack{0 \le i_1 \le 1 \\ 0 \le i_2, i_3, \dots, i_k \le k \\ i_1 + 2i_2 + \cdots + k\ {}_k = k}} \frac{F_1{}^{i_1} F_2^{i_2} F_3^{i_3} \dots F_k^{i_k}}{i_2!\, i_3! \dots i_k!}$$

then

$$S_k = (-1)^k \sum_{1 \le j_1 < j_2 < \cdots < j_k \le n} a_{j_1} a_{j_2} \dots a_{j_k} \qquad k \le n$$

$$S_k = 0 \qquad k > n$$

Identities 3.9. (D form of The Girard-Newton Identities, by Chen Shuwen, 2020)
Let n and k be positive integers. Denote

$$P_k = a_1^k + a_2^k + \cdots + a_n^k$$

and

$$D_1 = -P_1$$

$$D_k = \left(P_1^k - P_k\right)/k \qquad k \text{ is prime number}$$

$$D_k = \left(P_1^k - P_k\right)/k + \sum_{j=1}^{h}(-1)^{f_j}\frac{1}{f_j}D_{k/f_j}^{f_j} \qquad k \text{ is composite number with all its factors } f_1, f_2, \ldots, f_h \text{, exclude 1 and itself.}$$

and

$$S_k = \sum_{\substack{0\le i_1,i_2,i_3,\ldots,i_k\le 1 \\ i_1+2i_2+\cdots+ki_k=k}} D_1^{i_1}D_2^{i_2}D_3^{i_3}\ldots D_k^{i_k}$$

then

$$S_k = (-1)^k\sum_{1\le j_1<j_2<\cdots<j_k\le n} a_{j_1}a_{j_2}\ldots a_{j_k} \qquad k \le n$$

$$S_k = 0 \qquad k > n$$

Example 3.4.
Denote

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k \qquad (k = 1,2,3,4,5,6,7)$$

then

(1) By Identities 3.6, we have

$$S_7 = -\frac{P_1^7}{5040} + \frac{1}{240}P_1^5P_2 - \frac{1}{48}P_1^3P_2^2 + \frac{1}{48}P_1P_2^3 - \frac{1}{72}P_1^4P_3 + \frac{1}{12}P_1^2P_2P_3 - \frac{1}{24}P_2^2P_3 - \frac{1}{18}P_1P_3^2 + \frac{1}{24}P_1^3P_4 - \frac{1}{8}P_1P_2P_4 + \frac{P_3P_4}{12} - \frac{1}{10}P_1^2P_5 + \frac{P_2P_5}{10} + \frac{P_1P_6}{6} - \frac{P_7}{7} = 0$$

(2) By Identities 3.7, we have

$$S_7 = \frac{Q_1^7}{7!} + \frac{Q_1^5}{5!}Q_2 + \frac{Q_1^3}{3!}\frac{Q_2^2}{2!} + Q_1\frac{{Q_2}^3}{3!} + \frac{Q_1^4}{4!}Q_3 + \frac{Q_1^2}{2!}Q_2Q_3 + \frac{{Q_2}^2}{2!}Q_3 + Q_1\frac{Q_3^2}{2!} + \frac{Q_1^3}{3!}Q_4 + Q_1Q_2Q_4 + Q_3Q_4 + \frac{Q_1^2}{2!}Q_5 + Q_2Q_5 + Q_1Q_6 + Q_7 = 0$$

here $Q_k = -P_k/k \quad (k = 1,2,3,4,5,6,7)$

(3) By Identities 3.8, we have

$$S_7 = \frac{F_2^3}{3!}F_1 + \frac{F_2^2}{2!}F_3 + \frac{F_3^2}{2!}F_1 + F_1F_2F_4 + F_3F_4 + F_2F_5 + F_1F_6 + F_7 = 0$$

here $F_1 = -P_1$, $F_k = \left(P_1^k - P_k\right)/k \quad (k = 2,3,4,5,6,7)$

(4) By Identities 3.9, we have

$$S_7 = D_1D_2D_4 + D_3D_4 + D_2D_5 + D_1D_6 + D_7 = 0$$

here $D_1 = -P_1$ and

$$D_k = \left(P_1^k - P_k\right)/k \qquad (k = 2,3,5,7)$$

$$D_4 = (P_1^4 - P_4)/4 + \frac{D_2^2}{2} \qquad \left(= \frac{3P_1^4}{8} - \frac{1}{4}P_1^2P_2 + \frac{P_2^2}{8} - \frac{P_4}{4}\right)$$

$$D_6 = (P_1^6 - P_6)/4 - \frac{D_2^3}{3} + \frac{D_3^2}{2} \qquad \left(= \frac{13P_1^6}{72} + \frac{1}{8}P_1^4P_2 - \frac{1}{8}P_1^2P_2^2 + \frac{P_2^3}{24} - \frac{1}{9}P_1^3P_3 + \frac{P_3^2}{18} - \frac{P_6}{6}\right)$$

Corollary 3.1. (Proof by Identities 3.6 or 3.7 or 3.8 or 3.9)

1. There is no non-trivial solution for

$$a_1^k + a_2^k + a_3^k + \cdots + a_n^k = b_1^k + b_2^k + b_3^k + \cdots + b_n^k \quad (k = 1, 2, 3, \ldots, n)$$

2. If

$$a_1^k + a_2^k + a_3^k + \cdots + a_n^k = b_1^k + b_2^k + b_3^k + \cdots + b_n^k \quad (k = 1, 2, 3, \ldots, n-1, n+1)$$

then

$$a_1 + a_2 + a_3 + \cdots + a_n = b_1 + b_2 + b_3 + \cdots + b_n = 0$$

Identities 3.10. (A generalization of the Girard-Newton Identities, by Chen Shuwen, 2017)

Let

$$\begin{cases} P_k = a_1^k + a_2^k + \cdots + a_n^k, & \text{for } k \neq 0 \\ P_0 = a_1 a_2 \ldots a_n \end{cases}$$

and

$$\begin{cases} S_1 = -P_1 \\ S_k = -(P_k + S_1 P_{k-1} + S_2 P_{k-2} + \cdots + S_{k-1} P_1)/k, & \text{for } k > 1 \\ S_0 = -P_0 \\ S_{-1} = P_0 P_{-1} \\ S_k = -(-P_0 P_k + S_{-1} P_{k+1} + S_{-2} P_{k+2} + \cdots + S_{k+1} P_{-1})/(-k), & \text{for } k < -1 \end{cases}$$

and

$$\begin{cases} T_k = (-1)^n S_{k-n}, & \text{for } k \leq -1 \\ T_0 = 1 + (-1)^n S_{-n} \\ T_k = S_k + (-1)^n S_{k-n}, & \text{for } 1 \leq k \leq n \\ T_k = S_k, & \text{for } k \geq n+1 \end{cases}$$

then

$$T_k = 0$$

Remark. Here we generalize The Girard-Newton Identities from the case of $k > 0$ to the case of any integer k. It is useful for solving the system of the below form

$$a_1^k + a_2^k + a_3^k + \cdots + a_n^k = b_1^k + b_2^k + b_3^k + \cdots + b_n^k \quad (k = k_1, k_2, k_3, \ldots, k_m)$$
$$a_1 a_2 \ldots a_n = b_1 b_2 \ldots b_n$$

We will discuss this system in Chapter 5.

Example (Chen Shuwen, 2019/2/5).

$$\frac{1}{-63} + \frac{1}{-45} + \frac{1}{8} + \frac{1}{16} + \frac{1}{84} = \frac{1}{-72} + \frac{1}{-36} + \frac{1}{7} + \frac{1}{21} + \frac{1}{80}$$
$$(-63) \times (-45) \times 8 \times 16 \times 84 = (-72) \times (-36) \times 7 \times 21 \times 80$$
$$(-63) + (-45) + 8 + 16 + 84 = (-72) + (-36) + 7 + 21 + 80$$
$$(-63)^2 + (-45)^2 + 8^2 + 16^2 + 84^2 = (-72)^2 + (-36)^2 + 7^2 + 21^2 + 80^2$$
$$(-63)^4 + (-45)^4 + 8^4 + 16^4 + 84^4 = (-72)^4 + (-36)^4 + 7^4 + 21^4 + 80^4$$

Identities 3.11. (A generalization of the Girard-Newton Identities, by Chen Shuwen, 1997, 2016, 2020)
Let n and k and be positive integers. Denote

$$P_{2k+1} = a_1^{2k+1} + a_2^{2k+1} + \cdots + a_n^{2k+1}$$

and

$$G_1 = -P_1$$
$$G_3 = (P_1^3 - P_3)/3$$
$$G_5 = (P_1^5 - P_5)/5$$
$$G_7 = (P_1^7 - P_7)/7$$
$$G_9 = (P_1^9 - P_9)/9 - (\frac{G_3^3}{3})$$
$$G_{11} = (P_1^{11} - P_{11})/11 - (G_3^2 G_5)$$
$$G_{13} = (P_1^{13} - P_{13})/13 - (G_3 G_5^2 + G_3^2 G_7)$$
$$G_{15} = (P_1^{15} - P_{15})/15 - (\frac{G_3^5}{5} + \frac{G_5^3}{3} + 2G_3G_5G_7 + G_3^2 G_9)$$
$$G_{17} = (P_1^{17} - P_{17})/17 - (G_3^4 G_5 + G_5^2 G_7 + G_3 G_7^2 + 2G_3G_5G_9 + G_3^2 G_{11})$$
$$\vdots$$

In general

$$G_{2k+1} = (P_1^{2k+1} - P_{2k+1})/(2k+1) - 1/(2k+1)\sum_{j=1}^{k-3}(2j+1)G_{2j+1}Z_{2k-2j}$$

where

$$Z_{2k} = \sum_{\substack{i_3, i_5, \ldots, i_{2k-3} \geq 0 \\ 3i_3 + 5i_5 + \cdots + (2k-3)i_{2k-3} = 2k}} \frac{(i_3 + i_5 + \cdots + i_{2k-3})!}{i_3!\, i_5! \ldots i_{2k-3}!} G_3^{i_3} G_5^{i_5} \ldots G_{2k-3}^{i_{2k-3}}$$

Let

$$X_{n,0} = \begin{cases} (-1)^{\frac{n(n+2)}{8}} \begin{vmatrix} G_{n-1} & G_{n-3} & G_{n-5} & \cdots & G_5 & G_3 & G_1 \\ G_{n+1} & G_{n-1} & G_{n-3} & \cdots & G_7 & G_5 & G_1^3 + G_3 \\ G_{n+3} & G_{n+1} & G_{n-1} & \cdots & G_9 & G_7 & G_1^5 + G_3 G_1^2 + G_5 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ G_{2n-3} & G_{2n-5} & G_{2n-7} & \cdots & G_{n+3} & G_{n+1} & G_1^{n-1} + G_3 G_1^{n-4} + \cdots + G_{n-3} G_1^2 + G_{n-1} \end{vmatrix}, & n \text{ is even} \\ (-1)^{\frac{n^2-1}{8}} \begin{vmatrix} G_{n-2} & G_{n-4} & G_{n-6} & \cdots & G_3 & 0 & 1 \\ G_n & G_{n-2} & G_{n-4} & \cdots & G_5 & G_3 & G_1^2 \\ G_{n+2} & G_n & G_{n-2} & \cdots & G_7 & G_5 & G_1^4 + G_3 G_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ G_{2n-3} & G_{2n-5} & G_{2n-7} & \cdots & G_{n+2} & G_n & G_1^{n-1} + G_3 G_1^{n-4} + \cdots + G_{n-4} G_1^3 + G_{n-2} G_1 \end{vmatrix}, & n \text{ is odd} \end{cases}$$

$$X_{n,2}$$

$$= \begin{cases} (-1)^{\frac{n(n+2)}{8}+1} \begin{vmatrix} G_{n+1} & G_{n-3} & G_{n-5} & \cdots & G_5 & G_3 & G_1 \\ G_{n+3} & G_{n-1} & G_{n-3} & \cdots & G_7 & G_5 & G_1^3 + G_3 \\ G_{n+5} & G_{n+1} & G_{n-1} & \cdots & G_9 & G_7 & G_1^5 + G_3 G_1^2 + G_5 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ G_{2n-1} & G_{2n-5} & G_{2n-7} & \cdots & G_{n+3} & G_{n+1} & G_1^{n-1} + G_3 G_1^{n-4} + \cdots + G_{n-3} G_1^2 + G_{n-1} \end{vmatrix}, & n \text{ is even} \\ (-1)^{\frac{n^2-1}{8}+1} \begin{vmatrix} G_n & G_{n-4} & G_{n-6} & \cdots & G_3 & 0 & 1 \\ G_{n+2} & G_{n-2} & G_{n-4} & \cdots & G_5 & G_3 & G_1^2 \\ G_{n+4} & G_n & G_{n-2} & \cdots & G_7 & G_5 & G_1^4 + G_3 G_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ G_{2n-1} & G_{2n-5} & G_{2n-7} & \cdots & G_{n+2} & G_n & G_1^{n-1} + G_3 G_1^{n-4} + \cdots + G_{n-4} G_1^3 + G_{n-2} G_1 \end{vmatrix}, & n \text{ is odd} \end{cases}$$

$$X_{n,n}=\begin{cases}(-1)^{\frac{n(n-2)}{8}}\begin{vmatrix}G_{n+1}&G_{n-1}&G_{n-3}&\cdots&G_5&G_3\\G_{n+3}&G_{n+1}&G_{n-1}&\cdots&G_7&G_5\\G_{n+5}&G_{n+3}&G_{n+1}&\cdots&G_9&G_7\\\vdots&\vdots&\vdots&&\vdots&\vdots\\G_{2n-1}&G_{2n-3}&G_{2n-5}&\cdots&G_{n+3}&G_{n+1}\end{vmatrix}, & n\text{ is even}\\ \\(-1)^{\frac{(n-2)^2-1}{8}}\begin{vmatrix}G_n&G_{n-2}&G_{n-4}&\cdots&G_3&0\\G_{n+2}&G_n&G_{n-2}&\cdots&G_5&G_3\\G_{n+4}&G_{n+2}&G_n&\cdots&G_7&G_5\\\vdots&\vdots&\vdots&&\vdots&\vdots\\G_{2n-1}&G_{2n-3}&G_{2n-5}&\cdots&G_{n+2}&G_n\end{vmatrix}, & n\text{ is odd}\end{cases}$$

$U_{1,3,5,\ldots,2n-3,2n+1}$

$$=\begin{cases}(-1)^{\frac{n(n+2)}{8}+1}\begin{vmatrix}G_{n+1}&G_{n-3}&G_{n-5}&\cdots&G_5&G_3&G_1\\G_{n+3}&G_{n-1}&G_{n-3}&\cdots&G_7&G_5&G_1^3+G_3\\G_{n+5}&G_{n+1}&G_{n-1}&\cdots&G_9&G_7&G_1^5+G_3G_1^2+G_5\\\vdots&\vdots&\vdots&&\vdots&\vdots&\vdots\\G_{2n-3}&G_{2n-7}&G_{2n-9}&\cdots&G_{n+1}&G_{n-1}&G_1^{n-3}+G_3G_1^{n-6}+\cdots+G_{n-5}G_1^2+G_{n-3}\\G_{2n+1}&G_{2n-3}&G_{2n-5}&\cdots&G_{n+5}&G_{n+3}&G_1^{n+1}+G_3G_1^{n-2}+\cdots+G_{n-1}G_1^2+G_{n+1}\end{vmatrix}, & n\text{ is even}\\ \\(-1)^{\frac{n^2-1}{8}+1}\begin{vmatrix}G_n&G_{n-4}&G_{n-6}&\cdots&G_3&0&1\\G_{n+2}&G_{n-2}&G_{n-4}&\cdots&G_5&G_3&G_1^2\\G_{n+4}&G_n&G_{n-2}&\cdots&G_7&G_5&G_1^4+G_3G_1\\\vdots&\vdots&\vdots&&\vdots&\vdots&\vdots\\G_{2n-3}&G_{2n-7}&G_{2n-9}&\cdots&G_n&G_{n-2}&G_1^{n-3}+G_3G_1^{n-6}+\cdots+G_{n-6}G_1^3+G_{n-4}G_1\\G_{2n+1}&G_{2n-3}&G_{2n-5}&\cdots&G_{n+4}&G_{n+2}&G_1^{n+1}+G_3G_1^{n-2}+\cdots+G_{n-2}G_1^3+G_nG_1\end{vmatrix}, & n\text{ is odd}\end{cases}$$

then

$$X_{n,0}=\prod_{1\le i<j\le n}(a_i+a_j)$$

$$X_{n,2}=\prod_{1\le i<j\le n}(a_i+a_j)\sum_{1\le i<j\le n}a_ia_j$$

$$X_{n,n}=\prod_{1\le i<j\le n}(a_i+a_j)\prod_{1\le i\le n}(a_1+a_2+\cdots+a_n-a_i)$$

$$U_{1,3,5,\ldots,2n-3,2n+1}=\prod_{1\le i<j\le n}(a_i+a_j)\sum_{1\le i<j\le n}a_ia_j\left(\sum_{1\le i<j\le n}a_ia_j+\sum_{1\le i\le n}a_i^2\right)$$

and then

$$X_{n,2}/X_{n,0}=\sum_{1\le i<j\le n}a_ia_j$$

$$X_{n,n}/X_{n,0}=\prod_{1\le i\le n}(a_1+a_2+\cdots+a_n-a_i)$$

$$G_1^4-4\,U_{1,3,5,\ldots,2n-3,2n+1}/X_{n,0}=\left(\sum_{1\le i\le n}a_i^2\right)^2$$

Remark. Here we generalize The Girard-Newton Identities from the case of positive integer k to the case of all k be positive odd integers. It is useful for solving the system of the below form

$$a_1^k+a_2^k+a_3^k+\cdots+a_n^k=b_1^k+b_2^k+b_3^k+\cdots+b_n^k\qquad(k=2k_1+1,2k_2+1,\ldots,2k_m+1)$$

In the next section, we will generalize it to the case of all k be odd integers (both positive and negative).

Example 3.5. (Chen Shuwen, 2019, 2020)

Let n and k and be positive integers. Denote

$$P_{2k+1} = a_1^{2k+1} + a_2^{2k+1} + \cdots + a_n^{2k+1}$$

and

$$G_1 = -P_1$$
$$G_3 = (P_1^3 - P_3)/3$$
$$G_5 = (P_1^5 - P_5)/5$$
$$G_7 = (P_1^7 - P_7)/7$$
$$G_9 = (P_1^9 - P_9)/9 - (\frac{G_3^3}{3})$$
$$G_{11} = (P_1^{11} - P_{11})/11 - (G_3^2 G_5)$$
$$G_{13} = (P_1^{13} - P_{13})/13 - (G_3 G_5^2 + G_3^2 G_7)$$
$$G_{15} = (P_1^{15} - P_{15})/15 - (\frac{G_3^5}{5} + \frac{G_5^3}{3} + 2G_3G_5G_7 + G_3^2 G_9)$$
$$G_{17} = (P_1^{17} - P_{17})/17 - (G_3^4 G_5 + G_5^2 G_7 + G_3 G_7^2 + 2G_3G_5G_9 + G_3^2 G_{11})$$
$$G_{19} = (P_1^{19} - P_{19})/19 - (2G_3^3 G_5^2 + G_3^4 G_7 + G_5 G_7^2 + G_5^2 G_9 + 2G_3G_7G_9 + 2G_3G_5G_{11} + G_3^2 G_{13})$$

then

$$X_{2,0} = -|\, G_1 \,| = \prod_{1\le i<j\le 2} (a_i + a_j) = (a_1 + a_2)$$

$$X_{3,0} = -\begin{vmatrix} 0 & 1 \\ G_3 & G_1^2 \end{vmatrix} = \prod_{1\le i<j\le 3} (a_i + a_j) = (a_1 + a_2)(a_1 + a_3)(a_2 + a_3)$$

$$X_{4,0} = -\begin{vmatrix} G_3 & G_1 \\ G_5 & G_1^3 + G_3 \end{vmatrix} = \prod_{1\le i<j\le 4} (a_i + a_j) = (a_1 + a_2)(a_1 + a_3)(a_2 + a_3)(a_1 + a_4)(a_2 + a_4)(a_3 + a_4)$$

$$X_{5,0} = -\begin{vmatrix} G_3 & 0 & 1 \\ G_5 & G_3 & G_1^2 \\ G_7 & G_5 & G_1^4 + G_3G_1 \end{vmatrix} = \prod_{1\le i<j\le 5} (a_i + a_j)$$

$$X_{6,0} = \begin{vmatrix} G_5 & G_3 & G_1 \\ G_7 & G_5 & G_1^3 + G_3 \\ G_9 & G_7 & G_1^5 + G_3G_1^2 + G_5 \end{vmatrix} = \prod_{1\le i<j\le 6} (a_i + a_j)$$

$$X_{7,0} = \begin{vmatrix} G_5 & G_3 & 0 & 1 \\ G_7 & G_5 & G_3 & G_1^2 \\ G_9 & G_7 & G_5 & G_1^4 + G_3G_1 \\ G_{11} & G_9 & G_7 & G_1^6 + G_3G_1^3 + G_5G_1 \end{vmatrix} = \prod_{1\le i<j\le 7} (a_i + a_j)$$

$$X_{8,0} = \begin{vmatrix} G_7 & G_5 & G_3 & G_1 \\ G_9 & G_7 & G_5 & G_1^3 + G_3 \\ G_{11} & G_9 & G_7 & G_1^5 + G_3G_1^2 + G_5 \\ G_{13} & G_{11} & G_9 & G_1^7 + G_3G_1^4 + G_5G_1^2 + G_7 \end{vmatrix} = \prod_{1\le i<j\le 8} (a_i + a_j)$$

$$X_{9,0} = \begin{vmatrix} G_7 & G_5 & G_3 & 0 & 1 \\ G_9 & G_7 & G_5 & G_3 & G_1^2 \\ G_{11} & G_9 & G_7 & G_5 & G_1^4 + G_3G_1 \\ G_{13} & G_{11} & G_9 & G_7 & G_1^6 + G_3G_1^3 + G_5G_1 \\ G_{15} & G_{13} & G_{11} & G_9 & G_1^8 + G_3G_1^5 + G_5G_1^3 + G_7G_1 \end{vmatrix} = \prod_{1\le i<j\le 9} (a_i + a_j)$$

$$X_{10,0}=-\begin{vmatrix} G_9 & G_7 & G_5 & G_3 & G_1 \\ G_{11} & G_9 & G_7 & G_5 & G_1^3+G_3 \\ G_{13} & G_{11} & G_9 & G_7 & G_1^5+G_3G_1^2+G_5 \\ G_{15} & G_{13} & G_{11} & G_9 & G_1^7+G_3G_1^4+G_5G_1^2+G_7 \\ G_{17} & G_{15} & G_{13} & G_{11} & G_1^9+G_3G_1^6+G_5G_1^4+G_7G_1^2+G_9 \end{vmatrix}=\prod_{1\le i<j\le 10}(a_i+a_j)$$

$$X_{11,0}=-\begin{vmatrix} G_9 & G_7 & G_5 & G_3 & 0 & 1 \\ G_{11} & G_9 & G_7 & G_5 & G_3 & G_1^2 \\ G_{13} & G_{11} & G_9 & G_7 & G_5 & G_1^4+G_3G_1 \\ G_{15} & G_{13} & G_{11} & G_9 & G_7 & G_1^6+G_3G_1^3+G_5G_1 \\ G_{17} & G_{15} & G_{13} & G_{11} & G_9 & G_1^8+G_3G_1^5+G_5G_1^3+G_7G_1 \\ G_{19} & G_{17} & G_{15} & G_{13} & G_{11} & G_1^{10}+G_3G_1^7+G_5G_1^5+G_7G_1^3+G_9G_1 \end{vmatrix}=\prod_{1\le i<j\le 11}(a_i+a_j)$$

$$X_{12,0}=-\begin{vmatrix} G_{11} & G_9 & G_7 & G_5 & G_3 & G_1 \\ G_{13} & G_{11} & G_9 & G_7 & G_5 & G_1^3+G_3 \\ G_{15} & G_{13} & G_{11} & G_9 & G_7 & G_1^5+G_3G_1^2+G_5 \\ G_{17} & G_{15} & G_{13} & G_{11} & G_9 & G_1^7+G_3G_1^4+G_5G_1^2+G_7 \\ G_{19} & G_{17} & G_{15} & G_{13} & G_{11} & G_1^9+G_3G_1^6+G_5G_1^4+G_7G_1^2+G_9 \\ G_{21} & G_{19} & G_{17} & G_{15} & G_{13} & G_1^{11}+G_3G_1^8+G_5G_1^6+G_7G_1^4+G_9G_1^2+G_{11} \end{vmatrix}=\prod_{1\le i<j\le 12}(a_i+a_j)$$

$$X_{13,0}=-\begin{vmatrix} G_{11} & G_9 & G_7 & G_5 & G_3 & 0 & 1 \\ G_{13} & G_{11} & G_9 & G_7 & G_5 & G_3 & G_1^2 \\ G_{15} & G_{13} & G_{11} & G_9 & G_7 & G_5 & G_1^4+G_3G_1 \\ G_{17} & G_{15} & G_{13} & G_{11} & G_9 & G_7 & G_1^6+G_3G_1^3+G_5G_1 \\ G_{19} & G_{17} & G_{15} & G_{13} & G_{11} & G_9 & G_1^8+G_3G_1^5+G_5G_1^3+G_7G_1 \\ G_{21} & G_{19} & G_{17} & G_{15} & G_{13} & G_{11} & G_1^{10}+G_3G_1^7+G_5G_1^5+G_7G_1^3+G_9G_1 \\ G_{23} & G_{21} & G_{19} & G_{17} & G_{15} & G_{13} & G_1^{12}+G_3G_1^9+G_5G_1^7+G_7G_1^5+G_9G_1^3+G_{11}G_1 \end{vmatrix}=\prod_{1\le i<j\le 13}(a_i+a_j)$$

$$X_{14,0}=\begin{vmatrix} G_{13} & G_{11} & G_9 & G_7 & G_5 & G_3 & G_1 \\ G_{15} & G_{13} & G_{11} & G_9 & G_7 & G_5 & G_1^3+G_3 \\ G_{17} & G_{15} & G_{13} & G_{11} & G_9 & G_7 & G_1^5+G_3G_1^2+G_5 \\ G_{19} & G_{17} & G_{15} & G_{13} & G_{11} & G_9 & G_1^7+G_3G_1^4+G_5G_1^2+G_7 \\ G_{21} & G_{19} & G_{17} & G_{15} & G_{13} & G_{11} & G_1^9+G_3G_1^6+G_5G_1^4+G_7G_1^2+G_9 \\ G_{23} & G_{21} & G_{19} & G_{17} & G_{15} & G_{13} & G_1^{11}+G_3G_1^8+G_5G_1^6+G_7G_1^4+G_9G_1^2+G_{11} \\ G_{25} & G_{23} & G_{21} & G_{19} & G_{17} & G_{15} & G_1^{13}+G_3G_1^{10}+G_5G_1^8+G_7G_1^6+G_9G_1^4+G_{11}G_1^2+G_{13} \end{vmatrix}$$
$$=\prod_{1\le i<j\le 14}(a_i+a_j)$$

$$X_{15,0}=\begin{vmatrix} G_{13} & G_{11} & G_9 & G_7 & G_5 & G_3 & 0 & 1 \\ G_{15} & G_{13} & G_{11} & G_9 & G_7 & G_5 & G_3 & G_1^2 \\ G_{17} & G_{15} & G_{13} & G_{11} & G_9 & G_7 & G_5 & G_1^4+G_3G_1 \\ G_{19} & G_{17} & G_{15} & G_{13} & G_{11} & G_9 & G_7 & G_1^6+G_3G_1^3+G_5G_1 \\ G_{21} & G_{19} & G_{17} & G_{15} & G_{13} & G_{11} & G_9 & G_1^8+G_3G_1^5+G_5G_1^3+G_7G_1 \\ G_{23} & G_{21} & G_{19} & G_{17} & G_{15} & G_{13} & G_{11} & G_1^{10}+G_3G_1^7+G_5G_1^5+G_7G_1^3+G_9G_1 \\ G_{25} & G_{23} & G_{21} & G_{19} & G_{17} & G_{15} & G_{13} & G_1^{12}+G_3G_1^9+G_5G_1^7+G_7G_1^5+G_9G_1^3+G_{11}G_1 \\ G_{27} & G_{25} & G_{23} & G_{21} & G_{19} & G_{17} & G_{15} & G_1^{14}+G_3G_1^{11}+G_5G_1^9+G_7G_1^7+G_9G_1^5+G_{11}G_1^3+G_{13}G_1 \end{vmatrix}$$
$$=\prod_{1\le i<j\le 15}(a_i+a_j)$$

Example 3.6. (Chen Shuwen, 2020)

Let

$$P_k = a_1^k + a_2^k \quad (k = 1,3)$$

and

$$G_1 = -P_1$$

$$G_3 = (P_1^3 - P_3)/3$$

then

$$G_1^2 + 2\frac{|G_3|}{|G_1|} = a_1^2 + a_2^2$$

Let

$$P_k = a_1^k + a_2^k \quad (k = 1,5)$$

and

$$G_1 = -P_1$$

$$G_5 = (P_1^5 - P_5)/5$$

then

$$G_1^4 + 4\frac{|G_5|}{|G_1|} = (a_1^2 + a_2^2)^2$$

Let

$$P_k = a_1^k + a_2^k + a_3^k \quad (k = 1,3,5)$$

and

$$G_1 = -P_1$$

$$G_3 = (P_1^3 - P_3)/3$$

$$G_5 = (P_1^5 - P_5)/5$$

then

$$G_1^2 + 2\frac{\begin{vmatrix} G_3 & 1 \\ G_5 & G_1^2 \end{vmatrix}}{\begin{vmatrix} 0 & 1 \\ G_3 & G_1^2 \end{vmatrix}} = a_1^2 + a_2^2 + a_3^2$$

Let

$$P_k = a_1^k + a_2^k + a_3^k \quad (k = 1,3,7)$$

and

$$G_1 = -P_1$$

$$G_3 = (P_1^3 - P_3)/3$$

$$G_7 = (P_1^7 - P_7)/7$$

then

$$G_1^4 + 4\frac{\begin{vmatrix} G_3 & 1 \\ G_7 & G_1^4 + G_1G_3 \end{vmatrix}}{\begin{vmatrix} 0 & 1 \\ G_3 & G_1^2 \end{vmatrix}} = (a_1^2 + a_2^2 + a_3^2)^2$$

Let

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k \quad (k = 1,3,5,7)$$

and

$$G_1 = -P_1$$

$$G_3 = (P_1^3 - P_3)/3$$

$$G_5 = (P_1^5 - P_5)/5$$

$$G_7 = (P_1^7 - P_7)/7$$

then

$$G_1^2 + 2\frac{\begin{vmatrix} G_5 & G_1 \\ G_7 & G_1^3 + G_3 \end{vmatrix}}{\begin{vmatrix} G_3 & G_1 \\ G_5 & G_1^3 + G_3 \end{vmatrix}} = a_1^2 + a_2^2 + a_3^2 + a_4^2$$

Let

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k \quad (k = 1,3,5,9)$$

and

$$G_1 = -P_1$$

$$G_3 = (P_1^3 - P_3)/3$$

$$G_5 = (P_1^5 - P_5)/5$$

$$G_9 = (P_1^9 - P_9)/9 - (G_3^3/3)$$

then

$$G_1^4 + 4\frac{\begin{vmatrix} G_5 & G_1 \\ G_9 & G_1^5 + G_1^2G_3 + G_5 \end{vmatrix}}{\begin{vmatrix} G_3 & G_1 \\ G_5 & G_1^3 + G_3 \end{vmatrix}} = (a_1^2 + a_2^2 + a_3^2 + a_4^2)^2$$

Let

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k \quad (k = 1,3,5,7,9)$$

and

$$G_1 = -P_1$$

$$G_3 = (P_1^3 - P_3)/3$$

$$G_5 = (P_1^5 - P_5)/5$$

$$G_7 = (P_1^7 - P_7)/7$$

$$G_9 = (P_1^9 - P_9)/9 - (G_3^3/3)$$

then

$$G_1^2 + 2\frac{\begin{vmatrix} G_5 & 0 & 1 \\ G_7 & G_3 & G_1^2 \\ G_9 & G_5 & G_1^4 + G_3G_1 \end{vmatrix}}{\begin{vmatrix} G_3 & 0 & 1 \\ G_5 & G_3 & G_1^2 \\ G_7 & G_5 & G_1^4 + G_3G_1 \end{vmatrix}} = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2$$

Let

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k \quad (k = 1,3,5,7,11)$$

and

$$G_1 = -P_1$$

$$G_3 = (P_1^3 - P_3)/3$$

$$G_5 = (P_1^5 - P_5)/5$$

$$G_7 = (P_1^7 - P_7)/7$$

$$G_{11} = (P_1^{11} - P_{11})/11 - (G_3^2G_5)$$

then

$$G_1^4 + 4\frac{\begin{vmatrix} G_5 & 0 & 1 \\ G_7 & G_3 & G_1^2 \\ G_{11} & G_7 & G_1^6 + G_3G_1^3 + G_5G_1 \end{vmatrix}}{\begin{vmatrix} G_3 & 0 & 1 \\ G_5 & G_3 & G_1^2 \\ G_7 & G_5 & G_1^4 + G_3G_1 \end{vmatrix}} = (a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2)^2$$

Corollary 3.2 (Chen Shuwen, 2020)

If

$$\sum_{i=1}^{n} a_i^k = \sum_{i=1}^{n} b_i^k \qquad (\, k = 1, 3, 5, \ldots, 2n-1\,)$$

then

$$\prod_{1\le i<j\le n} (a_i + a_j) = \prod_{1\le i<j\le n} (b_i + b_j)$$

Example 3.7.1 (Chen Shuwen, 1999, 2020)

Let

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k \qquad (\, k = 1, 3, 5, 7, 9, 11\,)$$

and

$$G_1 = -P_1$$
$$G_3 = (P_1^3 - P_3)/3$$
$$G_5 = (P_1^5 - P_5)/5$$
$$G_7 = (P_1^7 - P_7)/7$$
$$G_9 = (P_1^9 - P_9)/9 - (G_3^3/3)$$
$$G_{11} = (P_1^{11} - P_{11})/11 - (G_3^2 G_5)$$

then

$$G_1^2 + 2\frac{\begin{vmatrix} G_7 & G_3 & G_1 \\ G_9 & G_5 & G_1^3 + G_3 \\ G_{11} & G_7 & G_1^5 + G_3G_1^2 + G_5 \end{vmatrix}}{\begin{vmatrix} G_5 & G_3 & G_1 \\ G_7 & G_5 & G_1^3 + G_3 \\ G_9 & G_7 & G_1^5 + G_3G_1^2 + G_5 \end{vmatrix}} = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2$$

Example 3.7.2 (Chen Shuwen, 2020)

Let

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k \qquad (\, k = 1, 3, 5, 7, 9, 13\,)$$

and

$$G_1 = -P_1$$
$$G_3 = (P_1^3 - P_3)/3$$
$$G_5 = (P_1^5 - P_5)/5$$
$$G_7 = (P_1^7 - P_7)/7$$
$$G_9 = (P_1^9 - P_9)/9 - (G_3^3/3)$$
$$G_{13} = (P_1^{13} - P_{13})/13 - (G_3G_5^2 + G_3^2G_7)$$

then

$$G_1^4 + 4\frac{\begin{vmatrix} G_7 & G_3 & G_1 \\ G_9 & G_5 & G_1^3 + G_3 \\ G_{13} & G_9 & G_1^7 + G_3G_1^4 + G_5G_1^2 + G_7 \end{vmatrix}}{\begin{vmatrix} G_5 & G_3 & G_1 \\ G_7 & G_5 & G_1^3 + G_3 \\ G_9 & G_7 & G_1^5 + G_3G_1^2 + G_5 \end{vmatrix}} = (a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2)^2$$

Example 3.7.3 (Chen Shuwen, 2020)

Let

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k \qquad (k = -1, 1, 3, 5, 7, 9)$$

and

$$G_{-1} = P_{-1}$$
$$G_1 = -P_1$$
$$G_3 = (P_1^3 - P_3)/3$$
$$G_5 = (P_1^5 - P_5)/5$$
$$G_7 = (P_1^7 - P_7)/7$$
$$G_9 = (P_1^9 - P_9)/9 - (G_3^3/3)$$

then

$$G_1^2 + 2\frac{\begin{vmatrix} G_5(1+G_{-1}G_1) & G_1 & G_{-1} \\ G_7 & G_3 & G_1 \\ G_9 & G_5 & G_1^3 + G_3 \end{vmatrix}}{\begin{vmatrix} G_3(1+G_{-1}G_1) & G_1 & G_{-1} \\ G_5 & G_3 & G_1 \\ G_7 & G_5 & G_1^3 + G_3 \end{vmatrix}} = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2$$

Example 3.7.4 (Chen Shuwen, 2020)

Let

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k \qquad (k = -1, 1, 3, 5, 7, 11)$$

and

$$G_{-1} = P_{-1}$$
$$G_1 = -P_1$$
$$G_3 = (P_1^3 - P_3)/3$$
$$G_5 = (P_1^5 - P_5)/5$$
$$G_7 = (P_1^7 - P_7)/7$$
$$G_{11} = (P_1^{11} - P_{11})/11 - (G_3^2 G_5)$$

then

$$G_1^4 + 4\frac{\begin{vmatrix} G_5(1+G_{-1}G_1) & G_1 & G_{-1} \\ G_7 & G_3 & G_1 \\ G_{11} & G_7 & G_1^5 + G_3 G_1^2 + G_5 \end{vmatrix}}{\begin{vmatrix} G_3(1+G_{-1}G_1) & G_1 & G_{-1} \\ G_5 & G_3 & G_1 \\ G_7 & G_5 & G_1^3 + G_3 \end{vmatrix}} = (a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2)^2$$

Example 3.7.5 (Chen Shuwen, 2020)

Let

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k \qquad (k = -3, -1, 1, 3, 5, 7)$$

and

$$G_{-1} = P_{-1}$$
$$G_{-3} = -(P_{-1}^3 - P_{-3})/3$$
$$G_1 = -P_1$$
$$G_3 = (P_1^3 - P_3)/3$$

$G_5 = (P_1^5 - P_5)/5$
$G_7 = (P_1^7 - P_7)/7$

then

$$G_1^2 + 2\frac{\begin{vmatrix} G_3(1+G_{-1}G_1) & G_{-1} & G_{-3} \\ G_5 & G_1 & G_{-1} \\ G_7 - G_1^2G_5 & G_3 & G_1 \end{vmatrix}}{\begin{vmatrix} G_1 & G_{-1} & G_{-3} \\ G_3 & G_1 & G_{-1} \\ G_5 - G_1^2G_3 & G_3 & G_1 \end{vmatrix}} = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2$$

Example 3.7.6 (Chen Shuwen, 2020)

Let

$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k \quad (k = -3, -1, 1, 3, 5, 9)$

and

$G_{-1} = P_{-1}$
$G_{-3} = -(P_{-1}^3 - P_{-3})/3$
$G_1 = -P_1$
$G_3 = (P_1^3 - P_3)/3$
$G_5 = (P_1^5 - P_5)/5$
$G_9 = (P_1^9 - P_9)/9 - (G_3^3/3)$

then

$$G_1^4 + 4\frac{\begin{vmatrix} G_3(1+G_{-1}G_1) & G_{-1} & G_{-3} \\ G_5 & G_1 & G_{-1} \\ G_9 - G_1^4G_5 - G_1G_3G_5 & G_5 & G_1^3 + G_3 \end{vmatrix}}{\begin{vmatrix} G_1 & G_{-1} & G_{-3} \\ G_3 & G_1 & G_{-1} \\ G_5 - G_1^2G_3 & G_3 & G_1 \end{vmatrix}} = (a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2)^2$$

Example 3.7.7 (Chen Shuwen, 2020)

Let

$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k \quad (k = -5, -3, -1, 1, 3, 5)$

and

$G_{-1} = P_{-1}$
$G_{-3} = -(P_{-1}^3 - P_{-3})/3$
$G_{-5} = -(P_{-1}^5 - P_{-5})/5$
$G_1 = -P_1$
$G_3 = (P_1^3 - P_3)/3$
$G_5 = (P_1^5 - P_5)/5$

then

$$G_1^2 + 2\frac{\begin{vmatrix} G_1 & G_{-3} & G_{-5} - G_{-1}^2G_{-3} \\ G_3 & G_{-1} & G_{-3} \\ G_5 - G_1^2G_3 & G_1 & G_{-1} \end{vmatrix}}{\begin{vmatrix} G_{-1} & G_{-3} & G_{-5} - G_{-1}^2G_{-3} \\ G_1 & G_{-1} & G_{-3} \\ G_3 & G_1 & G_{-1} \end{vmatrix}} = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2$$

Example 3.7.8 (Chen Shuwen, 2020)

Let

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k \qquad (\, k = \ -5, -3, -1, 1, 3, 7 \,)$$

and

$$G_{-1} = P_{-1}$$
$$G_{-3} = -(P_{-1}^3 - P_{-3})/3$$
$$G_{-5} = -(P_{-1}^5 - P_{-5})/5$$
$$G_1 = -P_1$$
$$G_3 = (P_1^3 - P_3)/3$$
$$G_7 = (P_1^7 - P_7)/7$$

then

$$G_1^4 + 4\frac{\begin{vmatrix} G_1 & G_{-3} & G_{-5} - G_{-1}^2 G_{-3} \\ G_3 & G_{-1} & G_{-3} \\ G_7 - G_1^4 G_3 - G_1 G_3^2 & G_1^3 + G_3 & G_1(1 + G_{-1}G_1) \end{vmatrix}}{\begin{vmatrix} G_{-1} & G_{-3} & G_{-5} - G_{-1}^2 G_{-3} \\ G_1 & G_{-1} & G_{-3} \\ G_3 & G_1 & G_{-1} \end{vmatrix}} = (a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2)^2$$

Example 3.8.1 (Chen Shuwen, 2020)

Let

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k + a_7^k \qquad (\, k = 1, 3, 5, 7, 9, 11, 13 \,)$$

and

$$G_1 = -P_1$$
$$G_3 = (P_1^3 - P_3)/3$$
$$G_5 = (P_1^5 - P_5)/5$$
$$G_7 = (P_1^7 - P_7)/7$$
$$G_9 = (P_1^9 - P_9)/9 - (G_3^3/3)$$
$$G_{11} = (P_1^{11} - P_{11})/11 - (G_3^2 G_5)$$
$$G_{13} = (P_1^{13} - P_{13})/13 - (G_3 G_5^2 + G_3^2 G_7)$$

then

$$G_1^2 + 2\frac{\begin{vmatrix} G_7 & G_3 & 0 & 1 \\ G_9 & G_5 & G_3 & G_1^2 \\ G_{11} & G_7 & G_5 & G_1^4 + G_3 G_1 \\ G_{13} & G_9 & G_7 & G_1^6 + G_3 G_1^3 + G_5 G_1 \end{vmatrix}}{\begin{vmatrix} G_5 & G_3 & 0 & 1 \\ G_7 & G_5 & G_3 & G_1^2 \\ G_9 & G_7 & G_5 & G_1^4 + G_3 G_1 \\ G_{11} & G_9 & G_7 & G_1^6 + G_3 G_1^3 + G_5 G_1 \end{vmatrix}} = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2$$

Example 3.8.2 (Chen Shuwen, 2020)

Let

$$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k + a_7^k + a_8^k \qquad (\, k = 1, 3, 5, 7, 9, 11, 13, 15 \,)$$

and

$$G_1 = -P_1$$
$$G_3 = (P_1^3 - P_3)/3$$

$G_5 = (P_1^5 - P_5)/5$
$G_7 = (P_1^7 - P_7)/7$
$G_9 = (P_1^9 - P_9)/9 - (G_3^3/3)$
$G_{11} = (P_1^{11} - P_{11})/11 - (G_3^2 G_5)$
$G_{13} = (P_1^{13} - P_{13})/13 - (G_3 G_5^2 + G_3^2 G_7)$
$G_{15} = (P_1^{15} - P_{15})/15 - (G_3^5/5 + G_5^3/3 + 2G_3G_5G_7 + G_3^2 G_9)$

then

$$G_1^2 + 2\frac{\begin{vmatrix} G_9 & G_5 & G_3 & G_1 \\ G_{11} & G_7 & G_5 & G_1^3 + G_3 \\ G_{13} & G_9 & G_7 & G_1^5 + G_3G_1^2 + G_5 \\ G_{15} & G_{11} & G_9 & G_1^7 + G_3G_1^4 + G_5G_1^2 + G_7 \end{vmatrix}}{\begin{vmatrix} G_7 & G_5 & G_3 & G_1 \\ G_9 & G_7 & G_5 & G_1^3 + G_3 \\ G_{11} & G_9 & G_7 & G_1^5 + G_3G_1^2 + G_5 \\ G_{13} & G_{11} & G_9 & G_1^7 + G_3G_1^4 + G_5G_1^2 + G_7 \end{vmatrix}} = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 + a_8^2$$

Example 3.8.3 (Chen Shuwen, 2020)

Let

$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k + a_7^k + a_8^k + a_9^k \quad (k = 1, 3, 5, 7, 9, 11, 13, 15, 17)$

and

$G_1 = -P_1$
$G_3 = (P_1^3 - P_3)/3$
$G_5 = (P_1^5 - P_5)/5$
$G_7 = (P_1^7 - P_7)/7$
$G_9 = (P_1^9 - P_9)/9 - (G_3^3/3)$
$G_{11} = (P_1^{11} - P_{11})/11 - (G_3^2 G_5)$
$G_{13} = (P_1^{13} - P_{13})/13 - (G_3 G_5^2 + G_3^2 G_7)$
$G_{15} = (P_1^{15} - P_{15})/15 - (G_3^5/5 + G_5^3/3 + 2G_3G_5G_7 + G_3^2 G_9)$
$G_{17} = (P_1^{17} - P_{17})/17 - (G_3^4 G_5 + G_5^2 G_7 + G_3 G_7^2 + 2G_3G_5G_9 + G_3^2 G_{11})$

then

$$G_1^2 + 2\frac{\begin{vmatrix} G_9 & G_5 & G_3 & 0 & 1 \\ G_{11} & G_7 & G_5 & G_3 & G_1^2 \\ G_{13} & G_9 & G_7 & G_5 & G_1^4 + G_3G_1 \\ G_{15} & G_{11} & G_9 & G_7 & G_1^6 + G_3G_1^3 + G_5G_1 \\ G_{17} & G_{13} & G_{11} & G_9 & G_1^8 + G_3G_1^5 + G_5G_1^3 + G_7G_1 \end{vmatrix}}{\begin{vmatrix} G_7 & G_5 & G_3 & 0 & 1 \\ G_9 & G_7 & G_5 & G_3 & G_1^2 \\ G_{11} & G_9 & G_7 & G_5 & G_1^4 + G_3G_1 \\ G_{13} & G_{11} & G_9 & G_7 & G_1^6 + G_3G_1^3 + G_5G_1 \\ G_{15} & G_{13} & G_{11} & G_9 & G_1^8 + G_3G_1^5 + G_5G_1^3 + G_7G_1 \end{vmatrix}} = a_1^2 + a_2^2 + a_3^2 + \cdots + a_9^2$$

Example 3.8.4 (Chen Shuwen, 2020)

Let

$P_k = a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k + a_7^k + a_8^k + a_9^k + a_{10}^k \quad (k = 1, 3, 5, 7, 9, 11, 13, 15, 17, 19)$

and

$G_1 = -P_1$

$G_3 = (P_1^3 - P_3)/3$
$G_5 = (P_1^5 - P_5)/5$
$G_7 = (P_1^7 - P_7)/7$
$G_9 = (P_1^9 - P_9)/9 - (G_3^3/3)$
$G_{11} = (P_1^{11} - P_{11})/11 - (G_3^2 G_5)$
$G_{13} = (P_1^{13} - P_{13})/13 - (G_3 G_5^2 + G_3^2 G_7)$
$G_{15} = (P_1^{15} - P_{15})/15 - (G_3^5/5 + G_5^3/3 + 2G_3 G_5 G_7 + G_3^2 G_9)$
$G_{17} = (P_1^{17} - P_{17})/17 - (G_3^4 G_5 + G_5^2 G_7 + G_3 G_7^2 + 2G_3 G_5 G_9 + G_3^2 G_{11})$
$G_{19} = (P_1^{19} - P_{19})/19 - (2G_3^3 G_5^2 + G_3^4 G_7 + G_5 G_7^2 + G_5^2 G_9 + 2G_3 G_7 G_9 + 2G_3 G_5 G_{11} + G_3^2 G_{13})$

then

$$G_1^2 + 2\frac{\begin{vmatrix} G_{11} & G_7 & G_5 & G_3 & G_1 \\ G_{13} & G_9 & G_7 & G_5 & G_1^3 + G_3 \\ G_{15} & G_{11} & G_9 & G_7 & G_1^5 + G_3 G_1^2 + G_5 \\ G_{17} & G_{13} & G_{11} & G_9 & G_1^7 + G_3 G_1^4 + G_5 G_1^2 + G_7 \\ G_{19} & G_{15} & G_{13} & G_{11} & G_1^9 + G_3 G_1^6 + G_5 G_1^4 + G_7 G_1^2 + G_9 \end{vmatrix}}{\begin{vmatrix} G_9 & G_7 & G_5 & G_3 & G_1 \\ G_{11} & G_9 & G_7 & G_5 & G_1^3 + G_3 \\ G_{13} & G_{11} & G_9 & G_7 & G_1^5 + G_3 G_1^2 + G_5 \\ G_{15} & G_{13} & G_{11} & G_9 & G_1^7 + G_3 G_1^4 + G_5 G_1^2 + G_7 \\ G_{17} & G_{15} & G_{13} & G_{11} & G_1^9 + G_3 G_1^6 + G_5 G_1^4 + G_7 G_1^2 + G_9 \end{vmatrix}} = a_1^2 + a_2^2 + a_3^2 + \cdots + a_{10}^2$$

Identities 3.12. (The generalization of the Girard-Newton Identities, Chen Shuwen, 1997, 2017, 2020)

Let n and m be positive integers, k be integer. Denote

$$R_k = \begin{cases} a_1^k + a_2^k + \cdots + a_n^k - (b_1^k + b_2^k + \cdots + b_m^k) & \text{for } k \neq 0 \\ -\dfrac{b_1 b_2 \dots b_m}{a_1 a_2 \dots a_n} & \text{for } k = 0 \end{cases}$$

and let

$$\begin{cases} U_1 = R_1 \\ U_k = (R_k + U_1 R_{k-1} + U_2 R_{k-2} + \cdots + U_{k-1} R_1)/k, & \text{for } k > 1 \\ U_0 = R_0 \\ U_{-1} = R_0 R_{-1} \\ U_k = (R_0 R_k + U_{-1} R_{k+1} + U_{-2} R_{k+2} + \cdots + U_{k+1} R_{-1})/(-k), & \text{for } k < -1 \end{cases}$$

and let

$$S_k = \gamma_k + \delta_k \qquad \text{where} \quad \gamma_k = \begin{cases} 0 & (k < 0) \\ 1 & (k = 0) \\ U_k & (k > 0) \end{cases}, \quad \delta_k = \begin{cases} (-1)^{m-n} U_{k-m+n} & (k \leq m-n) \\ 0 & (k > m-n) \end{cases}$$

and for $0 \leq t < n$, let

$$W_{nk+t} = \left| \begin{bmatrix} D_k \\ D_{k+1} \\ D_{k+2} \\ \vdots \\ D_{k+n-t-1} \\ D_{k+n-t+1} \\ \vdots \\ D_{k+n} \\ D_{k+n+1} \end{bmatrix} \right|, \text{where } D_k = [\, S_k \quad S_{k-1} \quad S_{k-2} \quad \dots \quad S_{k-n+2} \quad S_{k-n+1} \,]$$

Then

$$W_{nm} = \prod_{i=1}^{n} \prod_{j=1}^{m} (a_i - b_j)$$

and

$$\begin{cases} \dfrac{W_{nk+1}}{W_{nk}} = a_1 + a_2 + \cdots + a_n \\ \dfrac{W_{nk+2}}{W_{nk}} = a_1 a_2 + a_1 a_3 + \cdots + a_{n-1} a_n \\ \dfrac{W_{nk+3}}{W_{nk}} = a_1 a_2 a_3 + a_1 a_2 a_4 + \cdots + a_{n-2} a_{n-1} a_n \\ \vdots \\ \dfrac{W_{nk+n}}{W_{nk}} = a_1 a_2 a_3 \dots a_n \end{cases}$$

Remark. Here we generalize The Girard-Newton Identities from the case of $k > 0$ and $m = 0$ to the case of any integer k and any positive integer m, n. It is useful for solving the Generalization of Prouhet-Tarry-Escott Problem and Equal sums of like powers system, which will be discussed in Chapter 5.

Example 3.9.1 (Chen Shuwen, 2017, 2019, 2020)

Let

$$\begin{cases} R_1 = a_1 + a_2 + a_3 + a_4 - b_1 - b_2 - b_3 - b_4 - b_5 - b_6 \\ R_2 = a_1^2 + a_2^2 + a_3^2 + a_4^2 - b_1^2 - b_2^2 - b_3^2 - b_4^2 - b_5^2 - b_6^2 \\ R_3 = a_1^3 + a_2^3 + a_3^3 + a_4^3 - b_1^3 - b_2^3 - b_3^3 - b_4^3 - b_5^3 - b_6^3 \\ R_4 = a_1^4 + a_2^4 + a_3^4 + a_4^4 - b_1^4 - b_2^4 - b_3^4 - b_4^4 - b_5^4 - b_6^4 \\ R_5 = a_1^5 + a_2^5 + a_3^5 + a_4^5 - b_1^5 - b_2^5 - b_3^5 - b_4^5 - b_5^5 - b_6^5 \\ R_6 = a_1^6 + a_2^6 + a_3^6 + a_4^6 - b_1^6 - b_2^6 - b_3^6 - b_4^6 - b_5^6 - b_6^6 \\ R_7 = a_1^7 + a_2^7 + a_3^7 + a_4^7 - b_1^7 - b_2^7 - b_3^7 - b_4^7 - b_5^7 - b_6^7 \\ R_8 = a_1^8 + a_2^8 + a_3^8 + a_4^8 - b_1^8 - b_2^8 - b_3^8 - b_4^8 - b_5^8 - b_6^8 \\ R_9 = a_1^9 + a_2^9 + a_3^9 + a_4^9 - b_1^9 - b_2^9 - b_3^9 - b_4^9 - b_5^9 - b_6^9 \\ R_{10} = a_1^{10} + a_2^{10} + a_3^{10} + a_4^{10} - b_1^{10} - b_2^{10} - b_3^{10} - b_4^{10} - b_5^{10} - b_6^{10} \end{cases}$$

And

$$\begin{cases} S_1 = (R_1)/1 \\ S_2 = (R_2 + S_1R_1)/2 \\ S_3 = (R_3 + S_1R_2 + S_2R_1)/3 \\ S_4 = (R_4 + S_1R_3 + S_2R_2 + S_3R_1)/4 \\ S_5 = (R_5 + S_1R_4 + S_2R_3 + S_3R_2 + S_4R_1)/5 \\ S_6 = (R_6 + S_1R_5 + S_2R_4 + S_3R_3 + S_4R_2 + S_5R_1)/6 \\ S_7 = (R_7 + S_1R_6 + S_2R_5 + S_3R_4 + S_4R_3 + S_5R_2 + S_6R_1)/7 \\ S_8 = (R_8 + S_1R_7 + S_2R_6 + S_3R_5 + S_4R_4 + S_5R_3 + S_6R_2 + S_7R_1)/8 \\ S_9 = (R_9 + S_1R_8 + S_2R_7 + S_3R_6 + S_4R_5 + S_5R_4 + S_6R_3 + S_7R_2 + S_8R_1)/9 \\ S_{10} = (R_{10} + S_1R_9 + S_2R_8 + S_3R_7 + S_4R_6 + S_5R_5 + S_6R_4 + S_7R_3 + S_8R_2 + S_9R_1)/10 \end{cases}$$

and

$$W_{24} = \begin{vmatrix} S_6 & S_5 & S_4 & S_3 \\ S_7 & S_6 & S_5 & S_4 \\ S_8 & S_7 & S_6 & S_5 \\ S_9 & S_8 & S_7 & S_6 \end{vmatrix} \quad W_{25} = \begin{vmatrix} S_6 & S_5 & S_4 & S_3 \\ S_7 & S_6 & S_5 & S_4 \\ S_8 & S_7 & S_6 & S_5 \\ S_{10} & S_9 & S_8 & S_7 \end{vmatrix} \quad W_{26} = \begin{vmatrix} S_6 & S_5 & S_4 & S_3 \\ S_7 & S_6 & S_5 & S_4 \\ S_9 & S_8 & S_7 & S_6 \\ S_{10} & S_9 & S_8 & S_7 \end{vmatrix}$$

$$W_{27} = \begin{vmatrix} S_6 & S_5 & S_4 & S_3 \\ S_8 & S_7 & S_6 & S_5 \\ S_9 & S_8 & S_7 & S_6 \\ S_{10} & S_9 & S_8 & S_7 \end{vmatrix} \quad W_{28} = \begin{vmatrix} S_7 & S_6 & S_5 & S_4 \\ S_8 & S_7 & S_6 & S_5 \\ S_9 & S_8 & S_7 & S_6 \\ S_{10} & S_9 & S_8 & S_7 \end{vmatrix}$$

Then

$$\frac{W_{25}}{W_{24}} = a_1 + a_2 + a_3 + a_4 \qquad \frac{W_{26}}{W_{24}} = a_1a_2 + a_1a_3 + a_2a_3 + a_1a_4 + a_2a_4 + a_3a_4$$

$$\frac{W_{27}}{W_{24}} = a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4 \qquad \frac{W_{28}}{W_{24}} = a_1a_2a_3a_4$$

Remark. This identities is for solving the Prouhet-Tarry-Escott Problem of the below case.

$$a_0^k + a_1^k + a_2^k + a_3^k + a_4^k + a_5^k + a_6^k + a_7^k + a_8^k + a_9^k + a_{10}^k$$
$$= b_0^k + b_1^k + b_2^k + b_3^k + b_4^k + b_5^k + b_6^k + b_7^k + b_8^k + b_9^k + b_{10}^k$$
$$(k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$$

Remark. For the above Example 3.9

$$S_1 = a_1 + a_2 + a_3 + a_4 - b_1 - b_2 - b_3 - b_4 - b_5 - b_6$$

$$\begin{aligned} S_2 = {} & a_1^2 + a_1a_2 + a_2^2 + a_1a_3 + a_2a_3 + a_3^2 + a_1a_4 + a_2a_4 + a_3a_4 + a_4^2 - a_1b_1 - a_2b_1 - a_3b_1 - a_4b_1 - a_1b_2 \\ & - a_2b_2 - a_3b_2 - a_4b_2 + b_1b_2 - a_1b_3 - a_2b_3 - a_3b_3 - a_4b_3 + b_1b_3 + b_2b_3 - a_1b_4 - a_2b_4 \\ & - a_3b_4 - a_4b_4 + b_1b_4 + b_2b_4 + b_3b_4 - a_1b_5 - a_2b_5 - a_3b_5 - a_4b_5 + b_1b_5 + b_2b_5 + b_3b_5 \\ & + b_4b_5 - a_1b_6 - a_2b_6 - a_3b_6 - a_4b_6 + b_1b_6 + b_2b_6 + b_3b_6 + b_4b_6 + b_5b_6 \end{aligned}$$

$$\begin{aligned} S_3 = {} & a_1^3 + a_1^2a_2 + a_1a_2^2 + a_2^3 + a_1^2a_3 + a_1a_2a_3 + a_2^2a_3 + a_1a_3^2 + a_2a_3^2 + a_3^3 + a_1^2a_4 + a_1a_2a_4 + a_2^2a_4 + a_1a_3a_4 \\ & + \cdots + a_3b_5b_6 + a_4b_5b_6 - b_1b_5b_6 - b_2b_5b_6 - b_3b_5b_6 - b_4b_5b_6 \end{aligned}$$

… …

$$S_k = \sum_{\substack{0\le k_1,k_2,k_3,k_4\le k \\ 0\le k_5,k_6,k_7,k_8,k_9,k_{10}\le 1 \\ k_1+k_2+\cdots+k_9+k_{10}=k}} a_1^{k_1}a_2^{k_2}a_3^{k_3}a_4^{k_4}(-b_1)^{k_5}(-b_2)^{k_6}(-b_3)^{k_7}(-b_4)^{k_8}(-b_5)^{k_9}(-b_6)^{k_{10}}$$

and

$$\begin{aligned} W_{24} = {} & S_6^4 - 3S_5S_6^2S_7 + S_5^2S_7^2 + 2S_4S_6S_7^2 - S_3S_7^3 + 2S_5^2S_6S_8 - 2S_4S_6^2S_8 - 2S_4S_5S_7S_8 + 2S_3S_6S_7S_8 + S_4^2S_8^2 \\ & - S_3S_5S_8^2 - S_5^3S_9 + 2S_4S_5S_6S_9 - S_3S_6^2S_9 - S_4^2S_7S_9 + S_3S_5S_7S_9 \end{aligned}$$

$$\begin{aligned} W_{25} = {} & S_6^3S_7 - 2S_5S_6S_7^2 + S_4S_7^3 - S_5S_6^2S_8 + 2S_5^2S_7S_8 - S_3S_7^2S_8 - S_4S_5S_8^2 + S_3S_6S_8^2 + S_5^2S_6S_9 - S_4S_6^2S_9 \\ & - S_4S_5S_7S_9 + S_3S_6S_7S_9 + S_4^2S_8S_9 - S_3S_5S_8S_9 - S_5^3S_{10} + 2S_4S_5S_6S_{10} - S_3S_6^2S_{10} - S_4^2S_7S_{10} \\ & + S_3S_5S_7S_{10} \end{aligned}$$

$$\begin{aligned} W_{26} = {} & S_6^2S_7^2 - S_5S_7^3 - S_6^3S_8 + S_4S_7^2S_8 + S_4S_6S_8^2 - S_3S_7S_8^2 + S_5S_6^2S_9 + S_5^2S_7S_9 - 3S_4S_6S_7S_9 + S_3S_7^2S_9 - S_4S_5S_8S_9 \\ & + S_3S_6S_8S_9 + S_4^2S_9^2 - S_3S_5S_9^2 - S_5^2S_6S_{10} + S_4S_6^2S_{10} + S_4S_5S_7S_{10} - S_3S_6S_7S_{10} - S_4^2S_8S_{10} \\ & + S_3S_5S_8S_{10} \end{aligned}$$

$$\begin{aligned} W_{27} = {} & S_6S_7^3 - 2S_6^2S_7S_8 - S_5S_7^2S_8 + 2S_5S_6S_8^2 + S_4S_7S_8^2 - S_3S_8^3 + S_6^3S_9 - S_4S_7^2S_9 - S_5^2S_8S_9 - S_4S_6S_8S_9 \\ & + 2S_3S_7S_8S_9 + S_4S_5S_9^2 - S_3S_6S_9^2 - S_5S_6^2S_{10} + S_5^2S_7S_{10} + S_4S_6S_7S_{10} - S_3S_7^2S_{10} - S_4S_5S_8S_{10} \\ & + S_3S_6S_8S_{10} \end{aligned}$$

$$\begin{aligned} W_{28} = {} & S_7^4 - 3S_6S_7^2S_8 + S_6^2S_8^2 + 2S_5S_7S_8^2 - S_4S_8^3 + 2S_6^2S_7S_9 - 2S_5S_7^2S_9 - 2S_5S_6S_8S_9 + 2S_4S_7S_8S_9 + S_5^2S_9^2 \\ & - S_4S_6S_9^2 - S_6^3S_{10} + 2S_5S_6S_7S_{10} - S_4S_7^2S_{10} - S_5^2S_8S_{10} + S_4S_6S_8S_{10} \end{aligned}$$

and

$$\begin{aligned} W_{24} = {} & (a_1-b_1)(a_1-b_2)(a_1-b_3)(a_1-b_4)(a_1-b_5)(a_1-b_6) \\ & \times (a_2-b_1)(a_2-b_2)(a_2-b_3)(a_2-b_4)(a_2-b_5)(a_2-b_6) \\ & \times (a_3-b_1)(a_3-b_2)(a_3-b_3)(a_3-b_4)(a_3-b_5)(a_3-b_6) \\ & \times (a_4-b_1)(a_4-b_2)(a_4-b_3)(a_4-b_4)(a_4-b_5)(a_4-b_6) \end{aligned}$$

Example 3.9.2 (Chen Shuwen, 2017, 2019, 2020)

Let

$$\begin{cases} R_1 = a_1 + a_2 + a_3 + a_4 - b_1 - b_2 - b_3 - b_4 \\ R_2 = a_1^2 + a_2^2 + a_3^2 + a_4^2 - b_1^2 - b_2^2 - b_3^2 - b_4^2 \\ R_3 = a_1^3 + a_2^3 + a_3^3 + a_4^3 - b_1^3 - b_2^3 - b_3^3 - b_4^3 \\ R_4 = a_1^4 + a_2^4 + a_3^4 + a_4^4 - b_1^4 - b_2^4 - b_3^4 - b_4^4 \\ R_5 = a_1^5 + a_2^5 + a_3^5 + a_4^5 - b_1^5 - b_2^5 - b_3^5 - b_4^5 \\ R_6 = a_1^6 + a_2^6 + a_3^6 + a_4^6 - b_1^6 - b_2^6 - b_3^6 - b_4^6 \\ R_7 = a_1^7 + a_2^7 + a_3^7 + a_4^7 - b_1^7 - b_2^7 - b_3^7 - b_4^7 \\ R_8 = a_1^8 + a_2^8 + a_3^8 + a_4^8 - b_1^8 - b_2^8 - b_3^8 - b_4^8 \end{cases}$$

and

$$\begin{cases} S_1 = (R_1)/1 \\ S_2 = (R_2 + S_1R_1)/2 \\ S_3 = (R_3 + S_1R_2 + S_2R_1)/3 \\ S_4 = (R_4 + S_1R_3 + S_2R_2 + S_3R_1)/4 \\ S_5 = (R_5 + S_1R_4 + S_2R_3 + S_3R_2 + S_4R_1)/5 \\ S_6 = (R_6 + S_1R_5 + S_2R_4 + S_3R_3 + S_4R_2 + S_5R_1)/6 \\ S_7 = (R_7 + S_1R_6 + S_2R_5 + S_3R_4 + S_4R_3 + S_5R_2 + S_6R_1)/7 \\ S_8 = (R_8 + S_1R_7 + S_2R_6 + S_3R_5 + S_4R_4 + S_5R_3 + S_6R_2 + S_7R_1)/8 \end{cases}$$

and

$$W_{16} = \begin{vmatrix} S_4 & S_3 & S_2 & S_1 \\ S_5 & S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 & S_3 \\ S_7 & S_6 & S_5 & S_4 \end{vmatrix} \qquad W_{17} = \begin{vmatrix} S_4 & S_3 & S_2 & S_1 \\ S_5 & S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 & S_3 \\ S_8 & S_7 & S_6 & S_5 \end{vmatrix} \qquad W_{18} = \begin{vmatrix} S_4 & S_3 & S_2 & S_1 \\ S_5 & S_4 & S_3 & S_2 \\ S_7 & S_6 & S_5 & S_4 \\ S_8 & S_7 & S_6 & S_5 \end{vmatrix}$$

$$W_{19} = \begin{vmatrix} S_4 & S_3 & S_2 & S_1 \\ S_6 & S_5 & S_4 & S_3 \\ S_7 & S_6 & S_5 & S_4 \\ S_8 & S_7 & S_6 & S_5 \end{vmatrix} \qquad W_{20} = \begin{vmatrix} S_5 & S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 & S_3 \\ S_7 & S_6 & S_5 & S_4 \\ S_8 & S_7 & S_6 & S_5 \end{vmatrix}$$

Then

$$\begin{aligned} W_{16} = (a_1 - b_1)(a_1 - b_2)(a_1 - b_3)(a_1 - b_4) \\ \times (a_2 - b_1)(a_2 - b_2)(a_2 - b_3)(a_2 - b_4) \\ \times (a_3 - b_1)(a_3 - b_2)(a_3 - b_3)(a_3 - b_4) \\ \times (a_4 - b_1)(a_4 - b_2)(a_4 - b_3)(a_4 - b_4) \end{aligned}$$

and

$$\frac{W_{17}}{W_{16}} = a_1 + a_2 + a_3 + a_4 \qquad \frac{W_{18}}{W_{16}} = a_1a_2 + a_1a_3 + a_2a_3 + a_1a_4 + a_2a_4 + a_3a_4$$

$$\frac{W_{19}}{W_{16}} = a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4 \qquad \frac{W_{20}}{W_{16}} = a_1a_2a_3a_4$$

Example 3.9.3 (Chen Shuwen, 2017, 2019, 2020)

Let

$$\begin{cases} R_1 = a_1 + a_2 + a_3 + a_4 - b_1 - b_2 - b_3 - b_4 \\ R_2 = a_1^2 + a_2^2 + a_3^2 + a_4^2 - b_1^2 - b_2^2 - b_3^2 - b_4^2 \\ R_3 = a_1^3 + a_2^3 + a_3^3 + a_4^3 - b_1^3 - b_2^3 - b_3^3 - b_4^3 \\ R_4 = a_1^4 + a_2^4 + a_3^4 + a_4^4 - b_1^4 - b_2^4 - b_3^4 - b_4^4 \\ R_5 = a_1^5 + a_2^5 + a_3^5 + a_4^5 - b_1^5 - b_2^5 - b_3^5 - b_4^5 \\ R_6 = a_1^6 + a_2^6 + a_3^6 + a_4^6 - b_1^6 - b_2^6 - b_3^6 - b_4^6 \\ R_7 = a_1^7 + a_2^7 + a_3^7 + a_4^7 - b_1^7 - b_2^7 - b_3^7 - b_4^7 \\ R_8 = a_1^8 + a_2^8 + a_3^8 + a_4^8 - b_1^8 - b_2^8 - b_3^8 - b_4^8 \\ R_9 = a_1^9 + a_2^9 + a_3^9 + a_4^9 - b_1^9 - b_2^9 - b_3^9 - b_4^9 \\ R_0 = -\dfrac{b_1 b_2 b_3 b_4}{a_1 a_2 a_3 a_4} \\ R_{-1} = \dfrac{1}{a_1} + \dfrac{1}{a_2} + \dfrac{1}{a_3} + \dfrac{1}{a_4} - \dfrac{1}{b_1} - \dfrac{1}{b_2} - \dfrac{1}{b_3} - \dfrac{1}{b_4} \\ R_{-2} = \dfrac{1}{a_1^2} + \dfrac{1}{a_2^2} + \dfrac{1}{a_3^2} + \dfrac{1}{a_4^2} - \dfrac{1}{b_1^2} - \dfrac{1}{b_2^2} - \dfrac{1}{b_3^2} - \dfrac{1}{b_4^2} \\ R_{-3} = \dfrac{1}{a_1^3} + \dfrac{1}{a_2^3} + \dfrac{1}{a_3^3} + \dfrac{1}{a_4^3} - \dfrac{1}{b_1^3} - \dfrac{1}{b_2^3} - \dfrac{1}{b_3^3} - \dfrac{1}{b_4^3} \end{cases}$$

and

$$\begin{cases} S_1 = (R_1)/1 \\ S_2 = (R_2 + S_1R_1)/2 \\ S_3 = (R_3 + S_1R_2 + S_2R_1)/3 \\ S_4 = (R_4 + S_1R_3 + S_2R_2 + S_3R_1)/4 \\ S_5 = (R_5 + S_1R_4 + S_2R_3 + S_3R_2 + S_4R_1)/5 \\ S_6 = (R_6 + S_1R_5 + S_2R_4 + S_3R_3 + S_4R_2 + S_5R_1)/6 \\ S_7 = (R_7 + S_1R_6 + S_2R_5 + S_3R_4 + S_4R_3 + S_5R_2 + S_6R_1)/7 \\ S_8 = (R_8 + S_1R_7 + S_2R_6 + S_3R_5 + S_4R_4 + S_5R_3 + S_6R_2 + S_7R_1)/8 \\ S_9 = (R_9 + S_1R_8 + S_2R_7 + S_3R_6 + S_4R_5 + S_5R_4 + S_6R_3 + S_7R_2 + S_8R_1)/9 \\ S_0 = 1 + R_0 \\ S_{-1} = (R_0R_{-1})/1 \\ S_{-2} = (R_0R_{-2} + S_{-1}R_{-1})/2 \\ S_{-3} = (R_0R_{-3} + S_{-1}R_{-2} + S_{-2}R_{-1})/3 \end{cases}$$

and

$$W_8 = \begin{vmatrix} S_2 & S_1 & S_0 & S_{-1} \\ S_3 & S_2 & S_1 & S_0 \\ S_4 & S_3 & S_2 & S_1 \\ S_5 & S_4 & S_3 & S_2 \end{vmatrix} W_9 = \begin{vmatrix} S_2 & S_1 & S_0 & S_{-1} \\ S_3 & S_2 & S_1 & S_0 \\ S_4 & S_3 & S_2 & S_1 \\ S_6 & S_5 & S_4 & S_3 \end{vmatrix} W_{10} = \begin{vmatrix} S_2 & S_1 & S_0 & S_{-1} \\ S_3 & S_2 & S_1 & S_0 \\ S_5 & S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 & S_3 \end{vmatrix} W_{11} = \begin{vmatrix} S_2 & S_1 & S_0 & S_{-1} \\ S_4 & S_3 & S_2 & S_1 \\ S_5 & S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 & S_3 \end{vmatrix}$$

$$W_{12} = \begin{vmatrix} S_3 & S_2 & S_1 & S_0 \\ S_4 & S_3 & S_2 & S_1 \\ S_5 & S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 & S_3 \end{vmatrix} W_{13} = \begin{vmatrix} S_3 & S_2 & S_1 & S_0 \\ S_4 & S_3 & S_2 & S_1 \\ S_5 & S_4 & S_3 & S_2 \\ S_7 & S_6 & S_5 & S_4 \end{vmatrix} W_{14} = \begin{vmatrix} S_3 & S_2 & S_1 & S_0 \\ S_4 & S_3 & S_2 & S_1 \\ S_6 & S_5 & S_4 & S_3 \\ S_7 & S_6 & S_5 & S_4 \end{vmatrix} W_{15} = \begin{vmatrix} S_3 & S_2 & S_1 & S_0 \\ S_5 & S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 & S_3 \\ S_7 & S_6 & S_5 & S_4 \end{vmatrix}$$

$$W_{16}=\begin{vmatrix} S_4 & S_3 & S_2 & S_1 \\ S_5 & S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 & S_3 \\ S_7 & S_6 & S_5 & S_4 \end{vmatrix} W_{17}=\begin{vmatrix} S_4 & S_3 & S_2 & S_1 \\ S_5 & S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 & S_3 \\ S_8 & S_7 & S_6 & S_5 \end{vmatrix} W_{18}=\begin{vmatrix} S_4 & S_3 & S_2 & S_1 \\ S_5 & S_4 & S_3 & S_2 \\ S_7 & S_6 & S_5 & S_4 \\ S_8 & S_7 & S_6 & S_5 \end{vmatrix} W_{19}=\begin{vmatrix} S_4 & S_3 & S_2 & S_1 \\ S_6 & S_5 & S_4 & S_3 \\ S_7 & S_6 & S_5 & S_4 \\ S_8 & S_7 & S_6 & S_5 \end{vmatrix}$$

$$W_{20}=\begin{vmatrix} S_5 & S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 & S_3 \\ S_7 & S_6 & S_5 & S_4 \\ S_8 & S_7 & S_6 & S_5 \end{vmatrix} W_{21}=\begin{vmatrix} S_5 & S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 & S_3 \\ S_7 & S_6 & S_5 & S_4 \\ S_9 & S_8 & S_7 & S_6 \end{vmatrix} W_{22}=\begin{vmatrix} S_5 & S_4 & S_3 & S_2 \\ S_6 & S_5 & S_4 & S_3 \\ S_8 & S_7 & S_6 & S_5 \\ S_9 & S_8 & S_7 & S_6 \end{vmatrix} W_{23}=\begin{vmatrix} S_5 & S_4 & S_3 & S_2 \\ S_7 & S_6 & S_5 & S_4 \\ S_8 & S_7 & S_6 & S_5 \\ S_9 & S_8 & S_7 & S_6 \end{vmatrix}$$

Then

$$W_8/W_{16}=1/(a_1^2a_2^2a_3^2a_4^2)$$
$$W_9/W_{16}=(a_1+a_2+a_3+a_4)/(a_1^2a_2^2a_3^2a_4^2)$$
$$W_{10}/W_{16}=(a_1a_2+a_1a_3+a_2a_3+a_1a_4+a_2a_4+a_3a_4)/(a_1^2a_2^2a_3^2a_4^2)$$
$$W_{11}/W_{16}=(a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4)/(a_1^2a_2^2a_3^2a_4^2)$$
$$W_{12}/W_{16}=1/(a_1a_2a_3a_4)$$
$$W_{13}/W_{16}=(a_1+a_2+a_3+a_4)/(a_1a_2a_3a_4)$$
$$W_{14}/W_{16}=(a_1a_2+a_1a_3+a_2a_3+a_1a_4+a_2a_4+a_3a_4)/(a_1a_2a_3a_4)$$
$$W_{15}/W_{16}=(a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4)/(a_1a_2a_3a_4)$$
$$W_{16}=(a_1-b_1)(a_2-b_1)(a_3-b_1)(a_4-b_1)(a_1-b_2)(a_2-b_2)(a_3-b_2)(a_4-b_2)(a_1-b_3)(a_2-b_3)(a_3-b_3)(a_4-b_3)(a_1-b_4)(a_2-b_4)(a_3-b_4)(a_4-b_4)$$
$$W_{17}/W_{16}=a_1+a_2+a_3+a_4$$
$$W_{18}/W_{16}=a_1a_2+a_1a_3+a_2a_3+a_1a_4+a_2a_4+a_3a_4$$
$$W_{19}/W_{16}=a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4$$
$$W_{20}/W_{16}=a_1a_2a_3a_4$$
$$W_{21}/W_{16}=a_1a_2a_3a_4(a_1+a_2+a_3+a_4)$$
$$W_{22}/W_{16}=a_1a_2a_3a_4(a_1a_2+a_1a_3+a_2a_3+a_1a_4+a_2a_4+a_3a_4)$$
$$W_{23}/W_{16}=a_1a_2a_3a_4(a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4)$$

Identities 3.13. (The generalization of the Girard-Newton Identities, Chen Shuwen, 1997, 2019)

Assume

$$0\le n\le m$$

Let

$$\begin{cases} R_k=a_1^k+a_2^k+\cdots+a_n^k-(b_1^k+b_2^k+\cdots+b_m^k), & \text{for } k\ne 0 \\ R_0=-\dfrac{b_1b_2\ldots b_m}{a_1a_2\ldots a_n} \end{cases}$$

and

$$\begin{cases} T_1=R_1 \\ T_k=(R_k+T_1R_{k-1}+T_2R_{k-2}+\cdots+T_{k-1}R_1)/k, & \text{for } k>1 \\ T_0=R_0 \\ T_{-1}=R_0R_{-1} \\ T_k=(R_0R_k+T_{-1}R_{k+1}+T_{-2}R_{k+2}+\cdots+T_{k+1}R_{-1})/(-k), & \text{for } k<-1 \end{cases}$$

and

$$\begin{cases} S_k = (-1)^{m-n} T_{k-m+n}\,, & \text{for } k \le -1 \\ S_0 = 1 + (-1)^{m-n} T_{n-m} & \\ S_k = T_k + (-1)^{m-n} T_{k-m+n}\,, & \text{for } 1 \le k \le m-n \\ S_k = T_k\,, & \text{for } k \ge m-n+1 \end{cases}$$

and

$$\begin{cases} U_{0,k} = 1 \\ U_{1,k} = S_k \end{cases}$$

and for $2 \le h \le n$ and $1 \le t \le h-2$

$$\begin{cases} U_{h,hk} = \dfrac{U_{h-1,(h-1)k}^2 - U_{h-1,(h-1)(k-1)} U_{h-1,(h-1)(k+1)}}{U_{h-2,(h-2)k}} \\ U_{h,hk+t} = \dfrac{U_{h-1,(h-1)(k+1)} U_{h-1,(h-1)k+t-1} - U_{h-1,(h-1)(k+2)} U_{h-1,(h-1)k-h+t}}{U_{h-2,(h-2)(k+1)}} + \dfrac{U_{h,hk} U_{h-2,(h-2)(k+1)+t}}{U_{h-2,(h-2)(k+1)}} \\ U_{h,hk+h-1} = \dfrac{U_{h-1,(h-1)(k+1)} U_{h-1,(h-1)k+h-2} - U_{h-1,(h-1)(k+2)} U_{h-1,(h-1)k-1}}{U_{h-2,(h-2)(k+1)}} \end{cases}$$

and denote

$$W_{nk+t} \coloneqq U_{n,nk+t} \qquad (\text{or } W_t \coloneqq U_{n,t})$$

then for any k, we have

(1) When $n = 0$

$$S_k = 0$$

(2) When $0 < n \le m$

$$\begin{cases} W_{nm} = \prod_{i=1}^{n} \prod_{j=1}^{m} (a_i - b_j) \\ \dfrac{W_{nk+1}}{W_{nk}} = a_1 + a_2 + \cdots + a_n \\ \dfrac{W_{nk+2}}{W_{nk}} = a_1 a_2 + a_1 a_3 + \cdots + a_{n-1} a_n \\ \dfrac{W_{nk+3}}{W_{nk}} = a_1 a_2 a_3 + a_1 a_2 a_4 + \cdots + a_{n-2} a_{n-1} a_n \\ \dots\dots \\ \dfrac{W_{nk+n}}{W_{nk}} = a_1 a_2 a_3 \dots a_n \end{cases}$$

Remark. Identities 3.13 and Identities 3.12 are equivalent. Identities 3.12 looks more simple, but need to calculate determinants. Identities 3.13 is much more useful for searching the integer solutions of the Prouhet-Tarry-Escott Problem by using computers. This will be discussed in Chapter 5.

Chapter 4. The Generalization of Ramanujan Identity

To be completed. For some previous result, please see my below site.

http://eslpower.org/Identity.htm

Chapter 5. The Generalization of the Prouhet-Tarry-Escott Problem

To be completed. For plenty previous result, please see my below site.

http://eslpower.org

http://eslpower.org/eslp.htm

http://eslpower.org/TarryPrb.htm